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FINELY μ-HARMONIC FUNCTIONS OF A MARKOV PROCESS

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ABSTRACT. Let X be a Borel right process and m a fixed excessive measure. Given a finely open nearly Borel set G we define an operator Λ_G which we regard as an extension of the restriction to G of the generator of X. It maps functions on E to (locally) signed measures on G not charging m-semipolars. Given a locally smooth signed measure μ we define h to be (finely) μ-harmonic on G provided $(\Lambda_G + \mu)h = 0$ on G and denote the class of such h by $\mathcal{H}^\mu_f(G)$. Under mild conditions on X we show that $h \in \mathcal{H}^\mu_f(G)$ is equivalent to a local "Poisson" representation of h. We characterize $\mathcal{H}^\mu_f(G)$ by an analog of the mean value property under secondary assumptions. We obtain global Poisson type representations and study the Dirichlet problem for elements of $\mathcal{H}^\mu_f(G)$ under suitable finiteness hypotheses. The results take their nicest form when specialized to Hunt processes.

1. INTRODUCTION

In classical potential theory there are two equivalent, but at first glance rather different, approaches to defining a harmonic function. If h is a real valued function defined on an open set $G \subset \mathbb{R}^d$, then h is harmonic on G if (i) h is C^2 on G and $\Delta h = 0$ or (ii) h is continuous on G and satisfies the mean value property: if $B_r(x) = \{y : |y-x| < r\}$ is a ball with closure in G, $h(x) = \int_{S_r(x)} h(y)\sigma^{x,r}(dy)$ where $\sigma^{x,r}$ is normalized surface measure on $S_r(x) = \{y : |y-x| = r\}$. In extending the notion of harmonic function to Markov processes, the second approach immediately suggests itself because $\sigma^{x,r}$ has a direct probabilistic interpretation; it is the distribution of the place where a Brownian motion starting at x, first exits $B_r(x)$. A fundamental (Poisson) representation theorem for a harmonic function h on G is that if D is an open subset of G with $\bar{D}$ compact and $\bar{D} \subset G$, then $h(x) = \int_{\partial D} h(y)\sigma^x_D(dy)$ for $x \in D$ where σ^x_D is the harmonic measure of ∂D as viewed from x, or probabilistically the distribution of the place the Brownian motion starting from x exits D. There is a long history of using some variant of (ii) or the Poisson representation to define harmonic functions relative to a Markov process. See for example [M62], [Dy65], [BG68] or for more recent examples [Bo99] and [CS98].

However there is an inherent difficulty when the process has discontinuous paths and the exit measures σ^x_D are carried by all of D^c and not just ∂D. In particular h must be defined on G^c as well as G. The connection with (i) is more delicate. Although it is clear that the generator, Λ, of the underlying Markov process should

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replace the Laplacian Δ, there are "domain" problems in general. See [Dy65] for an early result relating (i) and (ii) and also [CZ95]. In [G99b] we introduced an "extended" generator, Λ, for a Markov process X and the motivation for this paper is to show the equivalence of $\Lambda h = 0$ and (local) Poisson type representations under mild hypotheses on X. Actually we consider a somewhat more general situation. We consider Λ_G—the extended generator Λ restricted to a finely open set G—and we consider μ-harmonic functions; that is h satisfying $(\Lambda_G + \mu)h = 0$ where μ is (locally) a signed measure. Again μ-harmonic functions have been studied by various authors in the literature. Classically $\Delta + \mu$ is often called the Schrödinger operator with potential μ. See [CZ95]. In addition [GH98] contains some very interesting results about μ-harmonic functions, although they do not use this terminology. Only the case, in our notation, $\mu \le 0$ is considered in [GH98].

We now give a rough outline of the paper. Section 2 introduces the precise assumptions on X and the basic notation. Throughout X is a transient Borel right process with state space E and m is a fixed σ-finite excessive measure that serves as background measure—Lebesgue measure in the classical situation. Section 3 begins with a review of the Revuz correspondence between the formal difference $\mu = \mu^+ - \mu^-$ of positive measures and continuous additive functionals (CAF's), $A = A^+ - A^-$. In particular smooth and locally smooth measures are defined. The most important results are Theorems 3.6 and 3.8. They have the form: given $\mu \ge 0$ with μ smooth on a finely open set G and A the positive CAF corresponding to μ, then there exist decompositions (G_n) of G with A having "good" finiteness properties on each G_n. In Section 4 the "extended" generator Λ_G is defined. It maps functions on E into measures $\mu = \mu^+ - \mu^-$ on G. Because X is transient, we are able to simplify somewhat the definition in [G99b].

Finally in Section 5 we define $\mathcal{H}_f^\mu(G)$ the class of finely μ-harmonic functions on a finely open nearly Borel set G for a locally smooth μ on E. Namely $h : E \to \mathbb{R}$ is in $\mathcal{H}_f^\mu(G)$ provided $h \in \mathcal{D}(\Lambda_G)$ and $(\Lambda_G + \mu)h = 0$ on G. It turns out that there are two local representations of $h \in \mathcal{H}_f^\mu(G)$ which reduce to the Poisson representation in the classical situation. The first, Theorem 5.3, states that $h \in \mathcal{H}_f^\mu(G)$ if and only if up to an m-polar set, G is a countable union of sets G_n such that if τ_n is the exit time of X from G_n, $h = P_{\tau_n} h + E^{\cdot} \int_0^{\tau_n} h(X_t)\, dA_t^n$ on G_n where A^n is the CAF corresponding to $1_{G_n}\mu$ and $P_{\tau_n} h = E^{\cdot}[h(X_{\tau_n})]$. It follows that h is finely continuous on each G_n. The second representation, Theorem 5.8, states that $h \in \mathcal{H}_f^\mu(G)$ if and only if with the previous notation, $h = E^{\cdot}[\exp(A^n_{\tau_n}) h(X_{\tau_n})]$ on G_n; however this is only proved assuming that X has no holding points or that $\mu^+(G) = 0$. Recall that $x \in E$ is a *holding point* provided $P^x(\tau_x > 0) = 1$ where $\tau_x = \inf\{t > 0 : X_t \ne x\}$. Although both of these representations reduce to $h = P_{\tau_n} h$ on G_n when $\mu = 0$, the second is the proper analog of the Poisson representation since it expresses h on G_n in terms of h on G_n^c. Section 6 contains some additional properties of finely μ-harmonic functions. Most important is Theorem 6.5 which is the true analog of the mean value property (ii). Also Theorem 6.8 presents the unique solution of a "Dirichlet" problem under suitable hypotheses. Since a finely μ-harmonic function h is defined on all of E, what we mean here by a "Dirichlet" problem is given a finely open set G and a function $g : G^c \to \mathbb{R}$ to find an element $h \in \mathcal{H}_f^\mu(G)$ which agrees with g on G^c. Hence the quotation marks. We refer the reader to Section 6 for the precise statements. In Section 7, it is shown that when m is a reference measure the exceptional m-polar set that appears in our definitions and theorems may be taken empty when μ is assumed to be locally strictly smooth as defined in Section 7. In addition the relationship between our interpretation of $(\Delta + \mu)h = 0$ and the interpretation in the sense of distributions in the classical situation is discussed.

We close this introduction with some words on notation. If $(F, \mathcal{F}, \mu)$ is a measure space, then we also use $\mathcal{F}$ to denote the class of all $\bar{\mathbb{R}} = [-\infty, \infty]$ valued $\mathcal{F}$ measurable functions. If $\mathcal{M} \subset \mathcal{F}$, then $b\mathcal{M}$ (resp. $p\mathcal{M}$) denotes the class of bounded (resp. $[0, \infty]$-valued) functions in $\mathcal{M}$. For $f \in p\mathcal{F}$ we shall use $\mu(f)$ to denote the integral $\int f d\mu$; similarly, if $D \in \mathcal{F}$ then $\mu(f; D)$ denotes $\int_D f d\mu$. We write $\mathcal{F}^*$ for the universal completion of $\mathcal{F}$; that is, $\mathcal{F}^* = \bigcap_\nu \mathcal{F}^\nu$, where $\mathcal{F}^\nu$ is the ν-completion of $\mathcal{F}$ and the intersection is over all finite (equivalently σ-finite) measures ν on $(F, \mathcal{F})$. If $(E, \mathcal{E})$ is a second measurable space and $K = K(x, dy)$ is a kernel from $(F, \mathcal{F})$ to $(E, \mathcal{E})$ (i.e., $F \ni x \mapsto K(x, A)$ is $\mathcal{F}$-measurable for each $A \in \mathcal{E}$ and $K(x, \cdot)$ is a measure on $(E, \mathcal{E})$ for each $x \in F$), then we write μK for the measure $A \mapsto \int_F \mu(dx) K(x, A)$ and Kf for the function $x \mapsto \int_E K(x, dy) f(y)$. The symbol ":=" stands for "is defined to be." Finally $\mathbb{R}$ (resp. $\mathbb{R}^+$) denotes the real numbers (resp. $[0, \infty[$) and $\mathcal{B}(\mathbb{R})$ (resp. $\mathcal{B}(\mathbb{R}^+)$) the corresponding Borel σ-algebras, while $\mathbb{Q}$ denotes the rationals. A reference (m.n) in the text refers to item m.n in section m. Due to the vagaries of LaTeX this might be a numbered display or the theorem, proposition, etc. numbered m.n.

2. Preliminaries

Throughout this paper $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, P^x)$ will denote the canonical realization of a Borel right Markov process with state space $(E, \mathcal{E})$. We shall use the standard notation for Markov processes as found, for example, in [BG68], [G90], [DM] and [Sh88]. Briefly, X is a strong Markov process with right continuous sample paths, the state space E (with Borel sets $\mathcal{E}$) is homeomorphic to a Borel subset of a compact metric space, and the transition semigroup $(P_t)_{t \geq 0}$ of X preserves the class $b\mathcal{E}$ of bounded $\mathcal{E}$-measurable functions. It follows that the resolvent operators $U^q := \int_0^\infty e^{-qt} P_t dt, q \geq 0$, also preserve Borel measurability. In the present situation q-excessive functions are nearly Borel and we let $\mathcal{E}^n$ denote the σ-algebra of nearly Borel subsets of E. *In the sequel, all named subsets of E are taken to be in $\mathcal{E}^n$ and all named functions are taken to be $\mathcal{E}^n$-measurable unless explicit mention is made to the contrary.*

We take Ω to be the canonical space of right continuous paths ω (with values in $E_\Delta := E \cup \{\Delta\}$) such that $\omega(t) = \Delta$ for all $t \geq \zeta(\omega) := \inf\{s : \omega(s) = \Delta\}$. The stopping time ζ is the *lifetime* of X and Δ is a cemetery state adjoined to E as an isolated point; Δ accounts for the possibility $P_t 1_E(x) < 1$ in that $P^x(\zeta < t) = 1 - P_t 1_E(x)$. The σ-algebras $\mathcal{F}_t$ and $\mathcal{F}$ are the usual completions of the σ-algebras $\mathcal{F}_t^\circ := \sigma\{X_s : 0 \leq s \leq t\}$ and $\mathcal{F}^\circ := \sigma\{X_s : s \geq 0\}$ generated by the coordinate maps $X_s : \omega \to \omega(s)$. The probability measure P^x is the law of X started at x, and for a measure μ on E, P^μ denotes $\int_E P^x(\cdot)\mu(dx)$. Finally, for $t \geq 0$, θ_t is the shift operator: $X_s \circ \theta_t = X_{s+t}$. We adhere to the convention that a function (resp. measure) on E (resp. $\mathcal{E}^*$) is extended to E_Δ by declaring its value at Δ (resp. $\{\Delta\}$) to be zero.

We fix once and for all an excessive measure m. Thus, m is a σ-finite measure on $(E, \mathcal{E}^*)$ and $mP_t \leq m$ for all $t > 0$. Since X is a right process, we then have $\lim_{t \to 0} mP_t = m$, setwise.

Recall that a set B is m-polar provided $P^m(T_B < \infty) = 0$, where $T_B := \inf\{t > 0 : X_t \in B\}$ denotes the hitting time of B. A property or statement $P(x)$ will be said to hold quasi-everywhere (q.e.), or for quasi-every $x \in E$, provided it holds for all x outside some m-polar subset of E. It would be more proper to use the term "m-quasi-everywhere," but since the measure m will remain fixed the abbreviation to "q.e." will cause no confusion. Similarly, the qualifier "a.e. m" will be abbreviated to "a.e." On the other hand, certain terms (e.g., polar) have a long-standing meaning without reference to a background measure, and so we shall use the more precise term "m-polar" to maintain the distinction. Notice that any finely open m-null set is m-polar. Consequently, any excessive function vanishing a.e. vanishes q.e. A set $B \subset E$ is m-semipolar provided it differs from a semipolar set by an m-polar set. It is known that B is m-semipolar if and only if

$$P^m(X_t \in B \text{ for uncountably many } t) = 0.$$

See [A73]. A set B is m-inessential provided it is m-polar and $E \setminus B$ is absorbing. According to [GS84, (6.12)] an m-polar set is contained in a Borel m-inessential set. Since m is excessive it follows that sets of potential zero are m-null. In particular m-polar and m-semipolar sets are m-null.

In order to keep technicalities at a minimum we shall assume throughout this paper that X is transient; that is,

Assumption 2.1. There exists a bounded strictly positive function $b \in \mathcal{E}^*$ such that $Ub = E^\cdot \int_0^\infty b(X_t)\,dt$ is bounded.

Of course the integral in t is only over the interval $[0, \zeta[$ since $X_t = \Delta$ if $t \geq \zeta$ and by convention $b(\Delta) = 0$. Replacing b by $U^1 b$ we may and shall suppose that $b \in \mathcal{E}^n$ and is finely continuous. It is known [G80] that 2.1 is equivalent to the apparently weaker assumption that there exists $b > 0$ with $Ub < \infty$.

For any $B \in E$, define

$$\tau_B = \tau(B) := \inf\{t > 0 : X_t \notin B\}.$$

τ_B is the *exit* time from B. Note that $\tau_B \leq \zeta$ and that $\tau_B = T_{B^c}$ if $T_{B^c} < \infty$ where $B^c = E \setminus B$. Of course if $D \subset E$, $\{T_D < \infty\} = \{T_D < \zeta\}$. Recall that all named sets are supposed to be nearly Borel unless explicitly stated otherwise. Define

$$B_p := \{x : P^x(\tau_B > 0) = 1\} = \{x : E^x(e^{-\tau(B)}) < 1\}.$$

Then B_p is finely open and is the set of *permanent* points for the multiplicative functional, $M_t := 1_{[0,\tau(B)[}(t)$.

Let $\mathcal{O}$ denote the class of finely open (nearly Borel) subsets of E. If $G \in \mathcal{O}$ then $G \subset G_p \subset \tilde{G}$ where "$\tilde{\ }$" denotes fine closure. Let $B^r = \{x : P^x(T_B = 0) = 1\}$ be the set of regular points of B. So $\tilde{B} = B \cup B^r$. If $G \in \mathcal{O}$ it is easy to check that $G_p \setminus G = G^c \setminus (G^c)^r$ where $G^c = E \setminus G$. Consequently $G_p \setminus G$ is semipolar. In analogy with regular open sets for the Dirichlet problem in classical potential theory we shall say that $G \in \mathcal{O}$ is *regular* if $G = G_p$. Note that $(G_p)_p = G_p$.

3. Additive Functionals

In this section we recall the definition and some basic properties of not necessarily increasing continuous additive functionals of X killed when it exits a finely open set. We first introduce the class of measures that will appear as Revuz measures of such a continuous additive functional (CAF).

Let $\mathcal{S}_0^+$ denote the class of σ-finite (positive) measures on $(E, \mathcal{E})$ that do not charge m-semipolars. Let $\mathcal{S}_0 := \mathcal{S}_0^+ - \mathcal{S}_0^+$ denote the class of all formal differences of elements of $\mathcal{S}_0^+$. Thus if $\mu_1, \mu_2 \in \mathcal{S}_0^+$, $\mu = (\mu_1, \mu_2)$ is formally $\mu = \mu_1 - \mu_2$. Equality in $\mathcal{S}_0$ is defined by $(\mu_1, \mu_2) = (\nu_1, \nu_2)$ provided $\mu_1 + \nu_2 = \mu_2 + \nu_1$. Then with the obvious definitions of addition and scalar multiplication $\mathcal{S}_0$ becomes a real vector space. We say that $\mu \in \mathcal{S}_0$ is *represented* by $(\mu_1, \mu_2) \in \mathcal{S}_0^+ \times \mathcal{S}_0^+$ when $\mu = (\mu_1, \mu_2)$. It is easy to check that each $\mu \in \mathcal{S}_0$ has a unique representation $\mu = (\mu^+, \mu^-)$ with $\mu^+ \perp \mu^-$. We then define $|\mu| = \mu^+ + \mu^-$. Note $|\mu| \in \mathcal{S}_0^+$. If f is finite a.e. $|\mu|$, then $f\mu := (f^+\mu^+ + f^-\mu^-, f^+\mu^- + f^-\mu^+) \in \mathcal{S}_0$ and checking carriers one sees that, in fact, $(f\mu)^+ = f^+\mu^+ + f^-\mu^-$ and $(f\mu)^- = f^+\mu^- + f^-\mu^+$ so that $|f\mu| = |f||\mu|$.

Let $G \in \mathcal{O}$ and $\tau = \tau_G$. Then the state space for (X, τ)—X killed when it exits G—is G_p defined in Section 2. Recall that m is our fixed excessive measure. The following definition is basic.

Definition 3.1. A continuous additive functional, A, of (X, τ) is a real valued process $A = A_t(\omega)$ defined on $0 \le t < \tau(\omega)$ if $\tau(\omega) > 0$ and for all $t \ge 0$ if $\tau(\omega) = 0$, for which there exists a defining set $\Lambda \in \mathcal{F}$ and an m-inessential set $N \subset G_p$—called an exceptional set for A—such that:

(i) $A_t 1_{\{t<\tau\}} \in \mathcal{F}_t$ for all t.
(ii) $P^x(\Lambda) = 1$ for $x \notin N$.
(iii) If $\omega \in \Lambda$ and $t < \tau(\omega)$, then $\theta_t\omega \in \Lambda$.
(iv) For $\omega \in \Lambda$, $t \to A_t(\omega)$ is continuous on $[0, \tau(\omega)[$ and of bounded variation on compact subintervals of $[0, \tau(\omega)[$.
(v) For all $\omega \in \Lambda$; $s \ge 0$, $t \ge 0$, $s + t < \tau(\omega)$ one has $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$.
(vi) $A_t(\omega) = 0$ for all t if $\tau(\omega) = 0$.

Note that if $\omega \in \Lambda$ and $\tau(\omega) > 0$ it follows from (v) that $A_0(\omega) = 0$. If A is increasing and we define for $\omega \in \Lambda$ and $t \ge \tau(\omega)$, $A_t(\omega) := \lim_{s\uparrow\tau(\omega)} A_s(\omega)$, then

$$A_{t+s}(\omega) = A_t(\omega) + 1_{[0,\tau(\omega)[}(t) A_s(\theta_t\omega) \tag{3.2}$$

for $\omega \in \Lambda$; $s, t \ge 0$. We denote the totality of all continuous additive functionals of (X, τ) by $\mathcal{A}(G)$ and by $\mathcal{A}^+(G)$ the increasing elements of $\mathcal{A}(G)$. If $A \in \mathcal{A}(G)$, $\omega \in \Lambda$ and $t < \tau(\omega)$ define $|A|_t(\omega)$ to be the total variation of $s \to A_s(\omega)$ on $[0, t]$. Then it is routine to check that $|A| \in \mathcal{A}^+(G)$ with the same defining and exceptional sets. Hence $A^+ := \frac{1}{2}[|A| + A]$ and $A^- := \frac{1}{2}[|A| - A]$ are in $\mathcal{A}^+(G)$ with the same defining and exceptional sets and $A = A^+ - A^-$. Two elements $A, B \in A(G)$ are equal provided they are m-equivalent; that is they have a common defining set Λ and a common exceptional set N such that $A_t(\omega) = B_t(\omega)$ for $\omega \in \Lambda$ and $0 \le t < \tau(\omega)$. The argument below (3.1) in [FG96] may be adapted to show that $A = B$ if and only if $P^m(A_t \ne B_t; t < \tau) = 0$ for all $t > 0$. Note we assume that N is m-inessential for X and not just for (X, τ). If A is a PCAF of X as defined in [FG96], then the restriction of A to $[0, \tau[$ is in $\mathcal{A}^+(G)$. Also if $A, B \in \mathcal{A}^+(G)$, then $A - B \in \mathcal{A}(G)$. Finally note that if $A^1, A^2, B^1, B^2 \in \mathcal{A}^+(G)$ then $A = A^1 - A^2$ equals $B = B^1 - B^2$ if and only if $A^1 + B^2 = A^2 + B^1$. Of course we are using m-equivalence as our definition of equality in $\mathcal{A}(G)$.

Definition 3.3. The Revuz measure associated with $A \in \mathcal{A}^+(G)$ is the measure ν_A defined by the formula

$$\nu_A(f) :=\uparrow \lim_{t\downarrow 0} E^m \frac{1}{t}\int_0^t f(X_s)\,dA_s, \qquad f \geq 0. \tag{3.4}$$

Of course the integral in (3.4) extends only over the interval $[0,\tau(\omega)[$ since the measure $dA_t(\omega)$ is carried by this interval. However it may be considered over $[0,\infty[$ since by convention we define $A_t(\omega) = \lim_{s\uparrow\tau(\omega)} A_s(\omega)$ for $t \geq \tau(\omega)$ when $A \in \mathcal{A}^+(G)$. See [FG88] for the fact that the limit in (3.4) exists. Since A is continuous a.s. P^m on $[0,\tau[$, it is clear that ν_A does not change m-semipolars, and since $\tau = 0$ a.s. P^x for $x \in E \setminus G_p$, that ν_A is carried by G_p and hence G since $G_p \setminus G$ is semipolar. It is also known that ν_A is σ-finite on G_p and hence on E. See [Re70, III.1]. It is a standard fact that ν_A determines A up to m-equivalence. Finally we have the classical uniqueness theorem: If $A, B \in \mathcal{A}^+(D)$ and if for some $\alpha \geq 0$,

$$E^{\cdot}\int_0^\tau e^{-\alpha t}dA_t = E^{\cdot}\int_0^\tau e^{-\alpha t}dB_t < \infty,$$

then $A = B$. For standard processes this is Theorem IV-(2.13) in [BG68]. The argument goes back to Meyer [M62] and works for continuous A whenever $t \to X_t$ has left limits a.s. on $]0,\zeta[$, and even for Borel right processes if one uses the Ray topology. The general case appears explicitly in [Sh88, (38.1)] along with existence for predictable A. See also [DM, VI-(69b)] for a general uniqueness theorem that easily implies the above result.

Definition 3.5. A (positive) measure ν on G is smooth provided it doesn't charge m-semipolars and there exists an increasing sequence (G_n) of finely open subsets of G such that

(i) $\nu(G_n) < \infty$ for each n.
(ii) $\tau_{G_n} \uparrow \tau_G$ a.s. P^x for q.e.x.

A sequence (G_n) of subsets of G satisfying (ii) of 3.5 is called a *nest* for G. It follows that $G \setminus \bigcup G_n$ is m-polar, hence ν null. Consequently ν is σ-finite on G. In particular if $\nu \in \mathcal{S}_0^+$ and $\nu(G) < \infty$, then $1_G\nu$ is smooth on G. The proof in [FG96] for the case $G = E$ is readily adapted to show that a measure ν on G is the Revuz measure of an $A \in \mathcal{A}^+(G)$ if and only if ν is smooth on G. Let $\mathcal{S}^+(G)$ denote the class of smooth measures on G. Then $A \leftrightarrow \nu_A$ is a bijection between $\mathcal{A}^+(G)$ and $\mathcal{S}^+(G)$. Of course when $E = G$ we drop it from our notation. Thus $\mathcal{S}^+$ denotes the smooth measures on E and $\mathcal{A}^+$ the PCAF's of X. Finally we identify a measure on G with a measure on E by extending it to be zero off G. Then $\mathcal{S}^+(G) \subset \mathcal{S}_0^+$. Clearly if $\mu \in \mathcal{S}^+(G)$ and $\nu \leq \mu$, then $\nu \in \mathcal{S}^+(G)$.

If $A \in \mathcal{A}(G)$ and $A = A^+ - A^-$ as defined above, we define $\nu_A := (\nu_{A^+}, \nu_{A^-}) \in \mathcal{S}_0$. If $A = A^1 - A^2$ is another decomposition of A into elements of $\mathcal{A}^+(G)$, then $\nu_A = (\nu_{A^1}, \nu_{A^2})$ in $\mathcal{S}_0$. Conversely if $\mu = (\mu_1, \mu_2) \in \mathcal{S}_0$ and μ_1 and μ_2 are in $\mathcal{S}^+(G)$, then $A = A^{\mu_1} - A^{\mu_2}$ is in $\mathcal{A}(G)$ and $\nu_A = \mu$. Let $\mathcal{S}(G)$ be those elements $\mu \in \mathcal{S}_0$ such that μ^+ and μ^- are in $\mathcal{S}^+(G)$ or equivalently $|\mu| \in \mathcal{S}^+(G)$. Then $\mathcal{S}(G)$ may be identified with $\mathcal{S}^+(G) - \mathcal{S}^+(G) \subset \mathcal{S}_0$ and $A \leftrightarrow \nu_A$ is now a bijection between $\mathcal{A}(G)$ and $\mathcal{S}(G)$. Moreover if $\mu \in \mathcal{S}(G)$ and $\nu \in \mathcal{S}_0$ with $|\nu| \leq |\mu|$ then $\nu \in \mathcal{S}(G)$. If $\mu \in \mathcal{S}(G)$ and $A \in \mathcal{A}(G)$ we write $\mu \leftrightarrow A$ when $\mu = \nu_A$.

Proposition 3.6. *Let $G \in \mathcal{O}$ and $A \in \mathcal{A}^+(G)$. Then there exists a nest (G_n) for G such that $\nu_A(G_n) < \infty$, $m(G_n) < \infty$ and letting $\tau = \tau_G$, $\tau_n = \tau(G_n)$ both of the functions $E^\cdot(\tau_n)$ and $E^\cdot(A_{\tau_n})$ are bounded on $E \setminus N$. Furthermore $G \setminus N = \bigcup G_n$ and $\tau_n \uparrow \tau$ a.s. P^x for $x \in E \setminus N$. Here N is an exceptional set for A.*

Proof. Deleting N from E it suffices to prove this when N is empty. Choose b with $0 < b \leq 1$, $Ub \leq 1$ and $m(b) < \infty$. Let $\varphi := E^\cdot \int_0^\tau e^{-A_t} b(X_t)\, dt$. Also define for $f \geq 0$,

$$U^\tau f := E^\cdot \int_0^\tau f(X_t)\, dt; \quad U_A^\tau f := E^\cdot \int_0^\tau f(X_t)\, dA_t. \tag{3.7}$$

Using the identity $1 = e^{-A_t} + e^{-A_t} \int_0^t e^{A_s}\, dA_s$ valid a.s. on $[0, \tau[$ it is readily verified that

$$1 \geq Ub \geq U^\tau b = \varphi + E^\cdot \int_0^\tau \varphi(X_t)\, dA_t = \varphi + U_A^\tau \varphi.$$

Hence φ is the difference of bounded excessive functions for (X, τ) and so is finely continuous and strictly positive on the finely open set G_p. Define $G_n = \{\varphi > \frac{1}{n}\} \cap G$. Then each $G_n \in \mathcal{O}$ and $G_n \uparrow G$. Now $U_A^\tau 1_{G_n} \leq n U_A^\tau \varphi \leq n U^\tau b$ and this implies that $\nu_A(G_n) \leq n \cdot m(b) < \infty$. See [Re70, p. 508]. Moreover

$$\begin{aligned} E^\cdot(A_{\tau_n}) = E^\cdot \int_0^{\tau_n} dA_t &= E^\cdot \int_0^{\tau_n} 1_{G_n}(X_t)\, dA_t \\ &\leq U_A^\tau 1_{G_n} \leq n U^\tau b \leq n, \end{aligned}$$

so $E^\cdot(A_{\tau_n})$ is bounded. If $\tau_n < \tau$, then $\varphi(X_{\tau_n}) \leq \frac{1}{n}$ and so

$$\begin{aligned} \frac{1}{n} &\geq E^\cdot[\varphi(X_{\tau_n}); \tau_n < \tau] = E^\cdot\left[e^{A_{\tau_n}} \int_{\tau_n}^\tau e^{-A_t} b(X_t)\, dt\right] \\ &\geq E^\cdot \int_{\tau_n}^\tau e^{-A_t} b(X_t)\, dt, \end{aligned}$$

which forces $\tau_n \uparrow \tau$ a.s. as $n \to \infty$ since $b > 0$ and $A_t < \infty$ for $0 \leq t < \tau$. To complete the proof just apply what has been proved to $B_t := A_t + (t \wedge \tau)$ and note that the Revuz measure of $t \to t \wedge \tau$ as an element of $\mathcal{A}^+(G)$ is $1_G \cdot m$. □

Notation. For typographical reasons we often write $A(t)$ for A_t or $X(t)$ for X_t. For example $A(\tau_n) = A_{\tau_n}$.

The next result complements 3.6. The assumption of no holding points is, perhaps, not too serious, but we only obtain a countable union and not a union of an increasing sequence. Recall the definition of holding point from Section 1. Let d be a metric on E compatible with the topology of E. Then for $H \subset E$, $\operatorname{diam} H = \sup_{x,y \in H} d(x, y)$ and $\bar{H}$ denotes the closure of H.

Theorem 3.8. *Let $G \in \mathcal{O}$ and $A \in \mathcal{A}^+(G)$. Suppose that X has no holding points. Then there exists a countable sequence (D_n) of finely open subsets of G with $G \setminus N = \bigcup D_n$ and such that each D_n has finite ν_A and m measure and the functions $E^\cdot(\tau(D_n))$ and $E^\cdot[\exp[A(\tau(D_n))]]$ are bounded on $E \setminus N$. As in 3.6, N is an exceptional set for A.*

Proof. Let G be one of the sets G_n in 3.6. Clearly it suffices to show that G is a countable union of sets D_n with the stated properties. Again, in the proof, we may suppose that N is empty. Let $\tau = \tau_G$. Then $E^{\cdot}(A_\tau)$ is bounded. Let $\mathcal{U}$ be a countable base of open sets for the topology of E. Fix $x \in G$ and choose a decreasing sequence $(H_k) \subset \mathcal{U}$ with diam $\bar{H}_k < \frac{1}{k}$ and $x \in \bigcap H_k$. Let $J_k = G \cap H_k$ and $\tau_k = \tau(J_k)$. Then $\tau_k \downarrow T \geq 0$. On $[0, \tau_k[$, $X_t \in J_k$ a.s. P^x. Consequently $X_t = x$ on $[0, T[$ a.s. P^x. But there are no holding points and so $P^x(T = 0) = 1$. Since $\tau_k \leq \tau$ and $E^x(A_\tau) < \infty$, it follows that $E^x(A_{\tau_k}) \downarrow 0$ as $k \to \infty$. Now for each $H \in \mathcal{U}$, let $J = H \cap G$ and define $\varphi_H = E^{\cdot}(A_{\tau(J)})$. Then φ_H is finely continuous and bounded on J. Fix η, $0 < \eta < 1$. Let $D(H) := J \cap \{\varphi_H < \eta\}$. Each $D(H)$ is finely open and the first part of the proof shows that $G = \bigcup\{D(H) : H \in \mathcal{U}\}$. Suppose $\sigma = \tau_{D(H)}$. Then $E^{\cdot}(A_\sigma) \leq \varphi_H \leq \eta$ on $\tilde{D}(H)$—the fine closure of $D(H)$. But as is well-known $E^{\cdot}(A_\sigma) \leq \eta < 1$ on $\tilde{D}(H)$ implies that $E^{\cdot}(\exp A_\sigma) \leq (1-\eta)^{-1} < \infty$ on $\tilde{D}(H)$. See for example [SS00]. If $x \notin \tilde{D}(H)$, $\sigma = 0$ a.s. P^x and so $E^{\cdot}(\exp(A_\sigma)) \leq (1-\eta)^{-1}$ everywhere. □

Remark. The proof shows that if $1 < \gamma < \infty$, the covering (D_n) in 3.8 may be chosen to satisfy $E^{\cdot}[\exp A_{\tau(D_n)}] \leq \gamma$. Just set $\eta = \gamma^{-1}(\gamma - 1)$ in the proof.

If $A \in \mathcal{A}(G)$, then $|A| \in \mathcal{A}^+(G)$ and so the result of 3.6 and 3.8 may be applied to $|A|$.

We close this section with some definitions that will be used in the sequel.

Definitions 3.9. Let $\mu \in \mathcal{S}_0$ and $G \in \mathcal{O}$.

(i) μ is locally smooth on G provided G is a countable union of sets $G_n \in \mathcal{O}$ with $\mu \in \mathcal{S}(G_n)$ for each n.

(ii) G is μ-integrable provided $|\mu|(G) < \infty$, $m(G) < \infty$ and if $A \leftrightarrow 1_G\mu$ both $E^{\cdot}(\tau_G)$ and $E^{\cdot}(|A|(\tau_G))$ are bounded on G.

(iii) G has a μ-integrable decomposition provided there exists a countable collection $(G_n) \subset \mathcal{O}$ of μ integrable subsets of G with $G \setminus \bigcup G_n$ being m-polar.

Remarks. If $G = \bigcup G_n$ with $(G_n) \subset \mathcal{O}$ and $|\mu|(G_n) < \infty$, then μ is locally smooth on G. Conversely if μ is locally smooth on G, then there exists $(G_n) \subset \mathcal{O}$ with $G_n \subset G$ and $|\mu|(G_n) < \infty$ for each n such that $G \setminus \bigcup G_n$ is m-polar. Here and in 3.9-iii one may assume that $G \setminus \bigcup G_n$ is m-inessential.

Notation 3.10. $\mathcal{S}_{loc}(G)$ denotes the class of $\mu \in \mathcal{S}_0$ which are locally smooth on G. Given $\mu \in \mathcal{S}_0$, $\mathcal{O}_\mu$ denotes the class of $G \in \mathcal{O}$ which are μ-integrable. We write $\mathcal{S}_{loc} = \mathcal{S}_{loc}(E)$.

Remarks 3.11. (i) If $\mu \in \mathcal{S}(G)$, then 3.6 implies that G has a μ-integrable decomposition with (G_n) being a nest for G.

(ii) $\mu \in \mathcal{S}_{loc}(G)$ then G has a μ-integrable decomposition.

(iii) If $\mu \in \mathcal{S}_{loc}(G)$ and X have no holding points, then G has a μ-integrable decomposition (G_n) such that if $A \leftrightarrow 1_{G_n}\mu$ then $E^{\cdot}(\exp[|A|(\tau_{G_n})])$ is bounded on G_n for each n.

(iv) Since the exceptional set N in 3.6 is an exceptional set for $A \in \mathcal{A}^+(G)$, if $\mu \in \mathcal{S}_{loc}(G)$ is such that $G = \bigcup G_n$ and $A^n \leftrightarrow 1_{G_n}\mu$ may be chosen without exceptional set for each n, then one may choose the μ-integrable decomposition (G_n) in (3.9–iii) so that $G = \bigcup G_n$. This is certainly the case if $\mu = 0$.

4. The Generator

In [G99b] we introduced an extended generator for X restricted to a finely open set. In the present paper we take advantage of our assumption that X is transient to modify (and simplify) the definition somewhat. We begin with some notation and a preliminary result before coming to the actual definition.

Let $G \in \mathcal{O}$ and let $A, B \in \mathcal{A}(G)$. If T is a stopping time with $T \le \tau_G$ and f is a function on E, define

$$P_T^A f := E^{\cdot}[e^{A_T} f(X_T)], \tag{4.1}$$

$$U_B^{A,T} f := E^{\cdot} \int_0^T e^{A_t} f(X_t)\, dB_t \tag{4.2}$$

whenever the integrals involved exist. If $A = 0$ we drop it in our notation writing merely $P_T f$ and U_B^T. If $B_t = t \wedge \tau_G$ we write $U^{A,T}$ in place of $U_B^{A,T}$. For example $U^T f = E^{\cdot} \int_0^T f(X_t)\, dt$. The following technical fact will be used in several places in the sequel.

Lemma 4.3. *Let $G \in \mathcal{O}$ and $A, B \in \mathcal{A}(G)$. Let $D \subset G$, $D \in \mathcal{O}$ and $\tau = \tau_D$. Let u and v be finite functions on E. Suppose that on D, $P_\tau^A |u|$ and $U_{|B|}^{A,\tau}|v|$ are finite (bounded) and $u = P_\tau^A u + U_B^{A,\tau} v$. If T is a stopping time with $T \le \tau$, then on D, $P_T^A|u|$ is finite (bounded) and $u = P_T^A u + U_B^{A,T} v$.*

Proof. On D, $U_{|B|}^{A,T}|v| \le U_{|B|}^{A,\tau}|v|$ is finite (bounded). Now since τ is a terminal time

$$P_\tau^A|u| \ge E^{\cdot}[e^{A_\tau}|u|(X_\tau); T < \tau] = E^{\cdot}[e^{A_T} P_\tau^A |u|(X_T); T < \tau].$$

But $X_T \in D$ on $\{T < \tau\}$ and on D, $|u| \le P_\tau^A|u| + U_{|B|}^{A,\tau}|v|$. Therefore

$$\begin{aligned} P_T^A|u| &\le E^{\cdot}[e^{A_T}(P_\tau^A|u|(X_T) + U_{|B|}^{A,\tau}|v|(X_T)) : T < \tau] + E^{\cdot}[e^{A_T}|u|(X_T); T = \tau] \\ &\le 2P_\tau^A|u| + E^{\cdot}\left[\int_T^\tau e^{A_t}|v|(X_t) d|B|_t, T < \tau\right]. \end{aligned}$$

Consequently $P_T^A|u|$ is finite (bounded) on D. Since all terms are finite one finds on D,

$$\begin{aligned} u &= P_\tau^A u + U_B^{A,T} v + E^{\cdot} \int_T^\tau e^{A_t} v(X_t)\, dB_t \\ &= P_\tau^A u + U_B^{A,T} v + E^{\cdot}[e^{A_T} U_B^{A,\tau} v(X_T), T < \tau]. \end{aligned}$$

But the expectation on the right side of the last display equals

$$E[e^{A_T}[u(X_T) - P_\tau^A u(X_T)]; T < \tau].$$

Moreover

$$\begin{aligned} P_\tau^A u &= E^{\cdot}[e^{A_\tau} u(X_\tau); T < \tau) + E^{\cdot}[e^{A_\tau} u(X_\tau); T = \tau] \\ &= E^{\cdot}[e^{A_T} P_\tau^A u(X_T); T < \tau] + E^{\cdot}[e^{A_T} u(X_T); T = \tau]. \end{aligned}$$

Combining these expressions we obtain $u = P_T^A u + U_B^{A,T} v$ on D. ☐

Remark. The two special cases $B = 0$ or $A = 0$ will be used most often in what follows.

Let $G \in \mathcal{O}$ be fixed. We are going to define an operator Λ_G that we regard as the "generator" of X restricted to G. Recall the definition 3.9 of a μ-integrable decomposition of G for $\mu \in \mathcal{S}_0$. Also recall that $\mathcal{S}_0(G) = \{\mu \in \mathcal{S}_0 : |\mu|(G^c) = 0\}$.

Definition 4.4. Let $G \in \mathcal{O}$. The domain $\mathcal{D}(\Lambda_G)$ of Λ_G consists of functions u defined on E which are finite and for which there exist $\mu \in \mathcal{S}_0$ and a μ-integrable decomposition (G_n) of G such that setting $\tau_n = \tau(G_n)$ for each n, u and $P_{\tau_n}|u|$ are bounded on G_n and

$$u = P_{\tau_n} u + E^{\cdot}(A^n_{\tau_n}) \qquad \text{on } G_n \tag{4.5}$$

where $A^n \leftrightarrow 1_{G_n}\mu$. If $u \in \mathcal{D}(\Lambda_G)$ then $\Lambda_G u = -1_G\mu$.

Remarks. One could just as well suppose that $\mu \in \mathcal{S}_0(G)$ in the definition since only the restriction of μ to G is relevant. In view of (3.11–ii) if $\mu \in \mathcal{S}_{loc}(G)$ then G has a μ-integrable decomposition.

Theorem 4.6. *Λ_G is a well-defined linear map from $\mathcal{D}(\Lambda_G)$ to $\mathcal{S}_0(G)$.*

Proof. We shall first show that Λ_G is well-defined. Let $u \in \mathcal{D}(\Lambda_G)$ with μ as in 4.4. Suppose that there also exist $\nu \in \mathcal{S}_0$ and a ν-integrable decomposition (H_n) of G such that if $\sigma_n = \tau_{H_n}$ one has on H_n that u and $P_{\sigma_n}|u|$ are bounded and $u = P_{\sigma_n}u + E^{\cdot}(B^n_{\sigma_n})$ where $B^n \leftrightarrow 1_{H_n}\mu$. Let $D_{n,k} = G_n \cap H_k$ and $\lambda = \tau(D_{n,k})$. It follows from 4.3 that on $D_{n,k}$, $P_\lambda|u|$ is bounded and $P_\lambda u + E^{\cdot}(A^n_\lambda) = u = P_\lambda u + E^{\cdot}(B^k_\lambda)$. Thus $E^{\cdot}[(A^n)^+_\lambda + (B^k)^-_\lambda] = E^{\cdot}[(A^n)^-_\lambda + (B^k)^+_\lambda]$ on $D_{n,k}$. Therefore $(A^n)^+ + (B^k)^- = (A^n)^- + (B^k)^+$ on $[0,\lambda[$ a.s. P^x for $x \in D_{n,k}$. Consequently Theorem 2.22 of [FG88] implies that $1_{D_{n,k}}\mu = 1_{D_{n,k}}\nu$. But $G \setminus \bigcup D_{n,k}$ is m-polar, and so $1_G\mu = 1_G\nu$. Thus Λ_G is well-defined.

Clearly if $\mu \in \mathcal{D}(\Lambda_G)$ and $\alpha \in \mathbb{R}$, then $\alpha u \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G(\alpha u) = \alpha\Lambda_G u$. Suppose u_1 and u_2 are in $\mathcal{D}(\Lambda_G)$ with $\Lambda_G u_j = -1_G\mu_j$, $j = 1, 2$. Let (G^j_n) be μ_j-integrable decompositions of G for $j = 1, 2$ such that the conditions in 4.4 hold. Then $G_{n,k} = G^1_n \cap G^2_k$ is a $\mu_1 + \mu_2$ integrable decomposition of G and because of 4.3, the conditions in 4.4 hold for $u_1 + u_2$ on each $G_{n,k}$. Therefore $u_1 + u_2 \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G(u_1 + u_2) = \Lambda_G u_1 + \Lambda_G u_2$, completing the proof of 4.6. □

Remarks 4.7. (i) If $u \in \mathcal{D}(\Lambda_G)$ and $H \in \mathcal{O}$ with $H \subset G$, then $u \in \mathcal{D}(\Lambda_H)$ and $\Lambda_H u = 1_H\Lambda_G u$. If $\mu \in \mathcal{S}_0$ and $(G_n) \subset \mathcal{O}$ with $u \in \mathcal{D}(\Lambda_{G_n})$ and $\Lambda_{G_n}u = -1_{G_n}\mu$ for each n, then $u \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G u = -1_G\mu$ where $G = \bigcup G_n$.

(ii) When $G = E$ we shall just write Λ for Λ_E. An equivalent description of Λ_G in martingale terms is given [G99b]. It follows from (3.6) and (4.4) of [G99b] that if $u \in \mathcal{D}(\Lambda_G)$, then u is quasi-finely continuous (q-f-continuous) on G in the sense that u is finely continuous on $G \setminus N$ where N is an m-inessential set. In fact it will be shown in the next section—see 5.2—that $P_{\tau_n}u$ is finely continuous on G_n and since $E^{\cdot}(A^n_{\tau_n})$ is also finely continuous on G_n so is u.

(iii) Theorem 2.22 of [FG88] used in the proof of 4.6 is a fairly deep result. However what is needed here is much simpler since X is Borel and the multiplicative functional in (2.22) of [FG88] used here is just $1_{[0,\lambda[}$. See (2.39a) in [FG88] in this connection.

(iv) In view of Proposition 5.2 in the next section and 4.3 if in definition 4.4 one only assumes that u and $P_{\tau_n}|u|$ are finite on each G_n, then it follows that $u \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G u = -1_G\mu$.

Here is an example to explain why we think of Λ_G as an extension of the restriction to G of the generator of X. Suppose $f \in b\mathcal{E}$ and $u := Uf$ is bounded. Then

u is in the domain of Λ_b, the generator of (P_t) acting on $\{f \in b\mathcal{E} : qU^q f \to f$ as $q \to \infty\}$ equipped with the sup norm and $\Lambda_b u = -f$. If $G \in \mathcal{O}$ and $\tau = \tau_G$, then $u = P_\tau u + E^\cdot \int_0^\tau f(X_s)\, ds$. Hence $u \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G u = -1_G fm$. In this sense Λ_G is an extension of the restriction of Λ_b to G. See the last paragraph of Section 4 of [G99b] for additional examples.

5. Harmonic Functions

In this section we fix $\mu \in \mathcal{S}_{loc}$ and we are going to define the notion of harmonicity relative to this fixed μ.

Definition 5.1. Let $h : E \to \mathbb{R}$ and $G \in \mathcal{O}$. Then h is finely μ-harmonic (μ-harmonic) on G provided h is quasi-finely continuous on G (G is open and h is continuous on G) with $h \in \mathcal{D}(\Lambda_G)$ and $(\Lambda_G + \mu)h = 0$ on G.

Remarks. Since h is finite, $h\mu \in \mathcal{S}_0$—in fact $h1_G\mu \in \mathcal{S}_{loc}$—and $(\Lambda_G + h)\mu = 0$ on G means $\Lambda_G\mu = -h1_G\mu$. Thus only the restriction of μ to G plays a role in the definition. In fact the assumption that h is quasi-finely continuous (abbreviated q-f-continuous from now on) is redundant as will follow from Proposition 5.3. The reason for including it is to contrast it with the condition for μ-harmonic.

Let $\mathcal{H}_f^\mu(G)$, resp. $\mathcal{H}^\mu(G)$, denote the class of finely μ-harmonic, resp. μ-harmonic, functions on G. We emphasize that elements of $\mathcal{H}_f^\mu(G)$ or $\mathcal{H}^\mu(G)$ are defined on all of E; although, of course, they may vanish on $E \setminus G$. If $(G_n) \subset \mathcal{O}$ with $h \in \mathcal{H}_f^\mu(G_n)$, resp. $\mathcal{H}^\mu(G_n)$, for each n, then $h \in \mathcal{H}_f^\mu(\bigcup G_n)$, resp. $\mathcal{H}^\mu(\bigcup G_n)$ in light of (4.7–i). Our main concern in this paper will be finely μ-harmonic functions, but we shall from time to time make comments about the specialization of our results to μ-harmonic functions. Clearly $\mathcal{H}_f^\mu(G)$ and $\mathcal{H}^\mu(G)$ are real vector spaces. If $\mu = 0$, we drop it from our notation. Thus $\mathcal{H}_f(G)$, resp. $\mathcal{H}(G)$, denotes the class of finely harmonic, resp. harmonic, functions on G. For example, $h \in \mathcal{H}_f(G)$ provided it is finite on E with $h \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G h = 0$. In this case (3.11–iv) and Proposition 5.2 imply that h is finely continuous on all of G.

Proposition 5.2. *Suppose that $f \geq 0$, $D \in \mathcal{O}$, $\tau = \tau_D$, $1_D\mu \in \mathcal{S}(D)$ and $A \leftrightarrow 1_D\mu$. Then $P_\tau^A f$ is finely continuous on D as a map from D to $[0, \infty]$. Here $e^{A(\tau)} = e^{A^+(\tau)} e^{-A^-(\tau)}$ with the usual convention that $0 \cdot \infty = 0$.*

Proof. Define $M_t := \exp(-A_t^-) 1_{[0,\tau[}(t)$. Then M is a decreasing multiplicative functional of X. If $f \geq 0$, let $Q_t f := E^\cdot(M_t f(X_t))$ be the semigroup of (X, M)—the M subprocess of X. If $A^- = 0$, in particular if $A = 0$, (X, M) is X killed when it exits D. The state space of (X, M) is $D_p \supset D$. One readily checks that if $f \geq 0$, then $P_\tau^A f$ is excessive for (X, M) and hence finely continuous on $D_p \supset D$. $\square$

The next proposition relates definition 5.1 to a type of Poisson representation.

Proposition 5.3. *Let $h : E \to \mathbb{R}$ and $G \in \mathcal{O}$.*

(i) Suppose that $1_G\mu \in \mathcal{S}(G)$ and $A \leftrightarrow 1_G\mu$. Let $\tau = \tau_G$ and assume that on G, $P_\tau|h| < \infty$, $U_{|A|}^\tau|h| < \infty$ and $h = P_\tau h + U_A^\tau h$. Then h is finely continuous on G and $h \in \mathcal{H}_f^\mu(G)$.

(ii) $h \in \mathcal{H}_f^\mu(G)$ if and only if there exists a μ-integrable decomposition (G_n) of G such that if $\tau(n) = \tau(G_n)$ and $A^n \leftrightarrow 1_{G_n}\mu$, then $h, P_{\tau(n)}|h|$ and $U_{|A^n|}^{\tau(n)}|h|$ are bounded on G_n and $h = P_{\tau(n)}h + U_{A^n}^{\tau(n)}h$ on G_n.

Proof. (i) By 5.2 with $A=0$, $P_\tau h = P_\tau h^+ - P_\tau h^-$ is the difference of finite finely continuous functions on G and hence is finely continuous on G. Also $U^\tau_{A^\pm} h^\pm$ for all possible choices of the signs is a finite (X,τ) excessive function. Consequently $U^\tau_A h$ is finite and finely continuous on G, and hence so is h. Now $1_G\mu \in \mathcal{S}(G)$ so by (3.11–i) there exists a μ-integrable decomposition (G_n) of G and since h and $P_\tau|h|$ are finite and finely continuous on G we may choose (G_n) so that they are bounded on each G_n. Then $U^{\tau(G_n)}_{|A|}|h|$ is bounded for each n and since $h = P_\tau h + U^\tau_A h$ on G it follows from 4.3 that $h = P_{\tau(n)} h + U^{\tau(n)}_A h$ on each G_n. Consequently $h \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G h = -1_G h\mu$; i.e. $h \in \mathcal{H}^\mu_f(G)$.

(ii) Suppose that $h \in \mathcal{H}^\mu_f(G)$. Then $h \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G h = -1_G h\mu$. Consequently there exists an $h\mu$-integrable decomposition (D_n) of G such that if $\tau_n = \tau(D_n)$ and if $B_n \in \mathcal{A}(D_n)$ with $B^n \leftrightarrow 1_{D_n} h\mu$ then on D_n, h, $P_{\tau_n}|h|$ and $E^\cdot(|B^n_{\tau_n}|)$ are bounded and $h = P_{\tau_n} h + E^\cdot(B^n_{\tau_n})$. But $\mu \in \mathcal{S}_{loc}$ and so there exists a μ-integrable decomposition (E_k) of E. Let $A^k \leftrightarrow 1_{E_k}\mu$. Then $(D_n \cap E_k)$ is a μ-integrable decomposition of G. Fix n and k for the moment and let $\sigma = \tau(D_n \cap E_k)$. Then $t \to B^n_{t\wedge\sigma}$ has Revuz measure $1_{D_n\cap E_k} h\mu$ as does $t \to \int_0^{t\wedge\sigma} h(X_s)\, dA^k_s$ and since h is bounded on $D_n \cap E_k$, $E^\cdot \int_0^\sigma |h|(X_s)\, d|A^k|_s$ is bounded on $D_n \cap E_k$. Combining this with 4.3 we see that $(D_n \cap E_k)$ has the properties asserted in (ii). Conversely it follows that $h \in \mathcal{D}(\Lambda_G)$ and $\Lambda_G h = -1_G h\mu$. As in the proof of (i), h is finely continuous on each G_n and hence q-f-continuous on G. Therefore $h \in \mathcal{H}^\mu_f(G)$. $\square$

Note that if $\mu = 0$, 5.3 becomes a Poisson type representation $h = P_{\tau_n} h$. We are going to obtain such a characterization when $\mu \neq 0$. The next result is the key step.

Proposition 5.4. *Let $D \in \mathcal{O}$, $\tau = \tau_D$ and $A \in \mathcal{A}(D)$ with $|A|_\tau < \infty$ a.s. Let $u : E \to \mathbb{R}$. Suppose that either $U^{|A|,\tau}_{|A|}[P_\tau|u| + |u|]$ is finite (bounded) on D or that $A^+ = 0$ and $U^\tau_{A^-}[P_\tau|u| + |u|]$ is finite (bounded) on D. Then on D, if either $P_\tau|u|$ or $P^A_\tau|u|$ is finite (bounded) so is the other and $u = P_\tau u + U^\tau_A u$ if and only if $u = P^A_\tau u$.*

Proof. Since D and τ are fixed in 5.4 we shall write $V_A = U^\tau_A$, $V^A_A = U^{A,\tau}_A$, etc. during the proof for notational simplicity. Because $|A|_\tau < \infty$, an integration by parts shows that

$$e^{A_\tau} + \int_0^\tau e^{A_t}\, dA^-_t = 1 + \int_0^\tau e^{A_t}\, dA^+_t.$$

Then since τ is a terminal time

$$\begin{aligned} P^A_\tau|u| + V^A_{A^-} P_\tau|u| &= E^\cdot\left[|u|(X_\tau)\Big[e^{A_\tau} + \int_0^\tau e^{A_t}\, dA^-_t\Big]\right] \\ &= P_\tau|u| + V^A_{A^+} P_\tau|u|. \end{aligned} \tag{5.5}$$

Suppose that $V^A_{|A|} P_\tau|u|$ is finite (bounded) on D. Then if either $P^A_\tau|u|$ or $P_\tau|u|$ is finite (bounded) so is the other and

$$P^A_\tau u = P_\tau u + V^A_A P_\tau u. \tag{5.6}$$

Now (5.6) in [G99b] implies that if f is any function with $V^{|A|}_{|A|}|f| < \infty$, then

$$V_A f + V^A_A V_A f = V^A_A f = V_A f + V_A V^A_A f. \tag{5.7}$$

This also is easily checked directly using the identities $e^{A_t} = 1 + \int_0^t e^{A_s}\, dA_s$ and $e^{A_t} = 1 + e^{A_t} \int_0^t e^{-A_s}\, dA_s$ on $[0, \tau[$. Thus if $u = P^A_\tau u$ on D we have using (5.6) and (5.7)

$$\begin{aligned} u = P^A_\tau u &= P_\tau u + V^A_A P_\tau u = P_\tau u + V_A[P_\tau u + V^A_A P_\tau u] \\ &= P_\tau u + V_A P^A_\tau u = P_\tau u + V_A u. \end{aligned}$$

Conversely, the subtraction being justified because $V^{|A|}_{|A|}|u| < \infty$,

$$P^A_\tau u = P_\tau u + V^A_A P_\tau u = P_\tau u + V^A_A[u - V_A u] = P_\tau u + V_A u = u.$$

Next suppose $A^+ = 0$ and $V_{A^-}(P_\tau|u| + |u|)$ is finite (bounded) on D. Then (5.5) becomes $P_\tau|u| = P^A_\tau|u| + V^A_{A^-} P_\tau|u|$. If $P_\tau|u|$ is finite (bounded) then so is $P^A_\tau|u|$ and since $V^A_{A^-} \le V_{A^-}$, if $P^A_\tau|u|$ is finite (bounded) so is $P_\tau|u|$. Let $B = A^-$. Then if $f \ge 0$, using the identities $1 = e^{-B_t} + \int_0^t e^{-B_s}\, dB_s$ and $1 = e^{-B_t} + e^{-B_t} \int_0^t e^{B_s}\, dB_s$ one obtains

$$V_B f = V^{-B}_B f + V^{-B}_B V_B f = V^{-B}_B f + V_B V^{-B}_B f.$$

Consequently for a general f if $V_B|f| < \infty$ on sees that (5.7) holds. The argument is now finished as before. $\square$

We come now to the main result of this section.

Theorem 5.8. *Let $G \in \mathcal{O}$ and $h : E \to \mathbb{R}$. Suppose either that X has no holding points or that $\mu^+(G) = 0$. Then $h \in \mathcal{H}^\mu_f(G)$ if and only if there exists a μ-integrable decomposition (G_n) of G such that if $\tau_n = \tau(G_n)$ and $A^n \leftrightarrow 1_{G_n}\mu$, then h and $P^{A^n}_{\tau_n}|h|$ are bounded on G_n and $h = P^{A^n}_{\tau_n} h$ on G_n.*

Proof. Suppose first that $h \in \mathcal{H}^\mu_f(G)$. Let (G_n) be a μ-integrable decomposition of G as in 5.3. If X has no holding points, then according to (3.11–iii) there exists a μ-integrable decomposition (E_n) of E such that if $B^n \leftrightarrow 1_{E_n}\mu$ and $\sigma_n = \tau(E_n)$, $E^\cdot \exp[|B^n|(\sigma_n)]$ is bounded on each E_n. Then by 4.3, $G_{n,k} := (E_n \cap G_k)$ is a μ-integrable decomposition of G such that if $\tau = \tau(G_{n,k})$ and $A \leftrightarrow 1_{G_{n,k}}\mu$, then $|h|$, $P_\tau|h|$ and $E^\cdot(e^{|A|(\tau)})$ are bounded on $G_{n,k}$. Consequently the hypotheses of 5.4 are satisfied because $U^{|A|,\tau}_{|A|} 1 = E^\cdot[e^{|A|(\tau)} - 1]$ is bounded on $G_{n,k}$. Combining 5.3, 4.3 and 5.4 it follows that $h = P^A_\tau h$ on $G_{n,k}$. If $\mu^+(G) = 0$, then in the decomposition (G_n), $A^{n+}_{\tau(G_n)} = 0$ for each n and so $U^{\tau_n}_{A^{n-}} 1 = E^\cdot(A^{n-}_{\tau(n)}) = E(|A^n|_{(\tau_n)})$ is bounded and the result follows as in the previous case. Conversely under the hypotheses of 5.8 one may assume as above that the decomposition in 5.8 satisfies the conditions in 5.4. Then 4.3, 5.4 and 5.3 combine to yield $h \in \mathcal{H}^\mu_f(G)$. $\square$

Corollary 5.9. *Let $G \in \mathcal{O}$, $\tau = \tau(G)$ and $1_G\mu \in \mathcal{S}(G)$ with $A \leftrightarrow 1_G\mu$. Suppose that either X has no holding points or $\mu^+(G) = 0$. If $h : E \to \mathbb{R}$ and on G, $P^A_\tau|h| < \infty$ and $h = P^A_\tau h$, then $h \in \mathcal{H}^\mu_f(G)$. Let $g : G^c \to \mathbb{R}$ satisfy $P^A_\tau|g| < \infty$ on G. Define $h := P^A_\tau g$ on G and $h = g$ on G^c. Then $h \in \mathcal{H}^\mu_f(G)$.*

Proof. It follows from 5.2, that h and $P^A_\tau|h|$ are finely continuous on G. Since $1_G\mu \in \mathcal{S}(G)$ there exists a μ-integrable decomposition (G_n) of G and we may suppose that h and $P^A_\tau|h|$ are bounded on each G_n because they are finely continuous on G. Let $\tau_n = \tau(G_n)$. Then 4.3 implies that $P^A_{\tau_n}|h|$ is bounded on each G_n and that $h = P^A_{\tau_n} h$ on G_n. Therefore (G_n) satisfies the hypotheses of 5.8 and hence $h \in \mathcal{H}^\mu_f(G)$. The second assertion now follows because $h = P^A_\tau g = P^A_\tau h$ on G since $g = h$ on G^c. $\square$

Remarks. The representation in 5.8 is the appropriate (local) analog of the classical Poisson representation. However the representation in 5.3 is often easier to work with and is valid without the additional hypotheses in 5.8. It expresses an $h \in \mathcal{H}_f^\mu(G)$ as locally the sum of an element in $\mathcal{H}_f(G)$, $P_{\tau(n)}h$ plus a "potential" $U_{A^n}^{\tau(n)}h$. Of course when $\mu = 0$ the two representations coincide. If $h \in \mathcal{H}^\mu(G)$, then h is continuous on G and hence on each G_n in 5.3 and 5.8. However the G_n cannot be assumed to be open without additional hypotheses. For example if $\mu = 0$, $\tau = \tau_G$ and (X, τ) excessive functions are lsc (lower semi-continuous), then the G_n may be chosen open, since φ in the proof of 3.6 is lsc. If $\mu = 0$, $G = E$ and b in 2.1 may be chosen so that Ub is lsc, then the G_n may be chosen open in characterizing $\mathcal{H}(E)$.

There is an important regularity property that elements of $\mathcal{H}_f^\mu(G)$ may enjoy. The next proposition will motivate the definition. Recall that the underlying Borel right process X is *special* provided the filtration $(\mathcal{F}_t)$ has no times of discontinuity; that is if (T_n) is an increasing sequence of stopping times with $T = \lim T_n$, then $\mathcal{F}_T = \vee\mathcal{F}_{T_n} := \sigma(\cup\mathcal{F}_{T_n})$. In particular a Hunt process is special.

Proposition 5.10. *Suppose u is finite, $D \in \mathcal{O}$, $1_{\mathcal{D}}\mu \in \mathcal{S}(D)$, $A \in \mathcal{A}(D)$ with $A \leftrightarrow 1_D\mu$. Let $\tau = \tau_D$. Assume that $P_\tau^A|u| < \infty$ and $u = P_\tau^A u$ on D. Then u is finely continuous on D and if T is a stopping time with $T \le \tau$, $P_T^A|u| \le P_\tau^A|u| < \infty$ on D. If, in addition, X is special then whenever (T_n) is an increasing sequence of stopping times with $T_n \uparrow T \le \tau$ one has $P_{T_n}^A u \to P_T^A u$ on D.*

Proof. The fine continuity of u on D is an immediate consequence of 5.2 Let $Y_\tau = e^{A_\tau}u(X_\tau)$. Note that $Y_\tau = 0$ on $\{\tau = \infty\}$ because of our convention that $u(\Delta) = 0$. Fix $x \in D$. Then $E^x|Y_\tau| = P_\tau^A|u|(x) < \infty$. Let T be a stopping time. Then

$$E^x[Y_\tau \mid \mathcal{F}_T] = e^{A_\tau}u(X_\tau)1_{\{\tau\le T\}} + E^x[e^{A_\tau}u(X_\tau); T < \tau \mid \mathcal{F}_T].$$

Since τ is a terminal time, this last conditional expectation equals

$$1_{\{T<\tau\}}e^{A_T}E^{X(T)}[e^{A_\tau}u(X_\tau)] = 1_{\{T<\tau)}e^{A_T}u(X_T)$$

because $X_T \in D$ on $\{T < \tau\}$ and $u = P_\tau^A u$ on D. Defining $Y_t = \exp(A_{t\wedge\tau})u(X_{t\wedge\tau})$, $t \ge 0$, it follows that (Y_t) is a P^x uniformly integrable (strong) martingale for $x \in D$. If T is a stopping time with $T \le \tau$, then on D

$$P_T^A|u| = E^{\cdot}(|Y_T|) \le E^{\cdot}(|Y_\tau|) = P_\tau^A|u| < \infty.$$

If (T_n) is an increasing sequence of stopping times with $T_n \uparrow T \le \tau$, then on D

$$Y_{T_n} = E^{\cdot}(Y_\tau \mid \mathcal{F}_{T_n}) \to E^{\cdot}(Y_\tau \mid \vee\mathcal{F}_{T_n}) = E^{\cdot}(Y_\tau \mid \mathcal{F}_T) = Y_T$$

provided X is special. But (Y_{T_n}) is P^x uniformly integrable and so $P_{T_n}^A u(x) \to P_T^A u(x)$ for $x \in D$. □

It will be convenient to introduce the following definition.

Definition 5.11. Let $f : E \to \mathbb{R}$ and $D \in \mathcal{O}$. Let $\tau = \tau_D$, $1_D\mu \in \mathcal{S}(D)$ and $A \leftrightarrow 1_D\mu$. Then f is μ-regular on D provided it is finely continuous on D, $P_T^A|f| < \infty$ if T is a stopping time with $T \le \tau$, and if (T_n) is a sequence of stopping times with limit $T \le \tau$ one has $P_{T_n}^A f \to P_T^A f$ on D.

If $\mu = 0$ we drop it from our notation and just say that f is regular on D. Thus 5.10 gives a sufficient condition for f to be μ-regular on D. Here is another. For its statement recall that X is quasi-left-continuous (qlc) provided $X_{T_n} \to X_T$ a.s. on $\{T < \zeta\}$ whenever (T_n) is an increasing sequence of stopping times with $T_n \uparrow T$.

Proposition 5.12. *Let $f : E \to \mathbb{R}$. Let $D \in \mathcal{O}$, $\tau = \tau_D$, $1_D\mu \in \mathcal{S}(D)$ and $A \leftrightarrow 1_D\mu$. Let $A^*_\tau = \sup\{A_t; t < \tau\}$. Note that $A^*_\tau \le A^+_\tau$. Suppose that f is bounded and continuous on $\bar{D}$ and that on D, $E^\cdot[e^{A^*_\tau}] < \infty$ and $P^A_\tau|f| < \infty$. If X is qlc and $\tau < \zeta$ a.s., then f is μ-regular on D.*

Proof. If T is a stopping time with $T \le \tau$, then

$$\begin{aligned} P^A_T|f|(x) &= E^\cdot[e^{A_T}|f|(X_T); T < \tau] + E^\cdot[e^{A_T}|f|(X_T); T = \tau] \\ &\le ME^\cdot e^{A^*(\tau)} + P^A_\tau|f| < \infty \end{aligned}$$

where M is a bound for f on D. Now suppose $T_n \uparrow T \le \tau < \zeta$ with the T_n being stopping times. Then $X(T_n) \to X(T)$ a.s. since X is qlc. Let $\Gamma = \{T_n < T \text{ for all } n\}$. Then a.s. on Γ, $X_T \in \bar{D}$ and $f(X(T_n)) \to f(X_T)$ boundedly. But $e^{A_{T_n}} \le e^{A^*_\tau}$ which is P^x-integrable for $x \in D$ and so $E^\cdot[e^{A(T_n)} f(X_{T_n}); \Gamma] \to E^\cdot[e^{A_T} f(X_T); \Gamma]$ on D. If M is a bound for f on D,

$$E^\cdot[e^{A(T_n)} f(X_{T_n}); T_n < T; \Gamma^c] \le ME^\cdot[e^{A^*_\tau}; T_n < T, \Gamma^c] \to 0,$$

since $\{T_n < T\} \cap \Gamma^c \downarrow \phi$ a.s. On the other hand $\{T_n = T\} \cap \Gamma^c \uparrow \Gamma^c$. Therefore

$$E^\cdot[e^{A(T_n)} f(X_{T_n}); T_n = T; \Gamma^c] \to E^\cdot[e^{A(T)} f(X_T); \Gamma^c]$$

by splitting the integral into an integral over $\{T < \tau\}$ where $|f(X_{T_n})| \le M$ and an integral over $\{T = \tau\}$ where $E^\cdot[e^{A(\tau)}|f(X_\tau)|] < \infty$. Combining these calculations we obtain $P^A_{T_n} f \to P^A_T f$ on D . ∎

Remarks 5.13. (i) It is easy to see that 5.12 is false even when $\mu = 0$ if the condition $\tau < \zeta$ a.s. is omitted. However if X is qlc on $[0, \infty[$, then $\tau < \infty$ a.s. suffices. This is the case if X is a Hunt process on a locally compact Hausdorff space with a countable base and Δ is the point at ∞.

(ii) The condition $P^A_T|f| < \infty$ on D for stopping times $T \le \tau$ in 5.11 is annoying. The proof of 5.12 shows that a sufficient condition for it is that f be bounded on D and that on D, $E^\cdot(e^{A^*_\tau}) < \infty$ and $P^A_\tau|f| < \infty$.

6. Representability and Harmonic Functions

Throughout this section as in Section 5, $\mu \in \mathcal{S}_{loc}$ is fixed.

Definition 6.1. Let $G \in \mathcal{O}$ and $\tau = \tau_G$. Suppose $1_G\mu \in \mathcal{S}(G)$ and let $A \leftrightarrow 1_G\mu$. Let $h : E \to \mathbb{R}$. Then h is representable on G provided $P^A_\tau|h| < \infty$ and $h = P^A_\tau h$ on G.

Notation. Let $\mathcal{R}(G)$ denote the collection of all h which are representable on G.

Remarks. It follows from 5.10 and 5.11 that if $h \in \mathcal{R}(G)$, then h is μ-regular on G provided X is special. If $h \in \mathcal{R}(G)$ and either X has no holding points or $\mu^+(G) = 0$, then $h \in \mathcal{H}^\mu_f(G)$. Also 4.3 implies that if $h \in \mathcal{R}(G)$ and $G_1 \subset G$ with $G_1 \in \mathcal{O}$, then $h \in \mathcal{R}(G_1)$.

There are two basic results in this section giving sufficient conditions for representability—Theorems 6.3 and 6.5. The next proposition is the key step in their proof and is of interest in its own right.

Proposition 6.2. *Let $G \in \mathcal{O}$, $\tau = \tau_G$, $1_G\mu \in \mathcal{S}(G)$ and $A \leftrightarrow 1_G\mu$. Suppose $h : E \to \mathbb{R}$ is μ-regular on G, in particular $P^A_\tau|h| < \infty$ on G. Assume that there exists a stopping time T such that $T \le \tau$, $T = 0$ on $\{X_0 \notin G\}$ and for each x in G, $h(x) = P^A_T h(x)$ and $P^x(T > 0) = 1$. Then $h \in \mathcal{R}(G)$, that is, $h = P^A_\tau h$ on G.*

Proof. For notational simplicity we shall abbreviate the exponential function by $e(\cdot)$ in this proof. For example $e(A_t) = \exp(A_t)$. Since h is μ-regular, $P_S^A|h| < \infty$ whenever S is a stopping time with $S \le \tau$. We are going to argue by transfinite induction. For each countable ordinal β we shall construct a stopping $T_\beta = T(\beta)$ such that:

(a) If $\alpha < \beta$, then $T_\alpha \le T_\beta \le \tau$ and $T_\alpha < T_\beta$ a.s. on $\{T_\alpha < \tau\}$.
(b) $h = P_{T(\beta)}^A h$ on G.

If $\beta = 1$, $T_1 = T$ satisfies (a) vacuously and (b). Suppose that β is a countable ordinal such that for each $\gamma < \beta$ we have constructed T_γ satisfying (a) and (b). If β has an immediate predecessor $\beta - 1$, let $R = T_{\beta-1}$ and set $T_\beta = R + T \circ \theta_R$. If $R < \tau$, $X_R \in G$ and so $T \circ \theta_R > 0$. Hence $R < T_\beta \le \tau$. If $R = \tau$. $X_R \notin G$ so $T \circ \theta_R = 0$. Therefore $T_\beta = R \le \tau$. Here and in what follows we omit the qualifying phrase "a.s." where it is clearly required. Thus (a) holds for T_β. For (b), $h = P_R^A h$ on G and so on G

$$\begin{aligned}
E^{\cdot}[e(A_{T(\beta)})h(X_{T(\beta)})] &= E^{\cdot}[e(A_R)e(A_T \circ \theta_R)h(X_T) \circ \theta_R] \\
&= E^{\cdot}[e(A_R)P_T^A h(X_R); R < \tau] + E^{\cdot}[e(A_{R+T\circ\theta_R})h(X_T) \circ \theta_R; R = \tau] \\
&= E^{\cdot}[e(A_R)h(X_R); R < \tau] + E^{\cdot}[e(A_R)h(X_R); R = \tau] \\
&= P_R^A h = h
\end{aligned}$$

where the third equality follows because $X_R \in G$ on $\{R < \tau\}$ and $X_R \notin G$ on $\{R = \tau\}$. Hence T_β satisfies (a) and (b).

Next suppose that β is a limit ordinal. Let $\gamma_n \uparrow \beta$. In view of (a), (T_{γ_n}) is increasing and $T_\beta := \lim_n T_{\gamma_n}$ satisfies (a). It remains to check that (b) holds for T_β. But it is evident that $h = P_{T_{\gamma_n}}^A h \to P_{T_\beta}^A h$ on G because h is μ-regular on G and $T_\beta \le \tau$. Consequently by transfinite induction there exists a stopping time T_β with properties (a) and (b) for each countable ordinal β.

Fix $x \in G$ and let $\varphi(\beta) = E^x[T_\beta(1 + T_\beta)^{-1}]$ for β a countable ordinal. Then $\gamma \le \beta$ implies $\varphi(\gamma) \le \varphi(\beta) \le 1$ and $\varphi(\gamma) < \varphi(\beta)$ if $P^x(T_\gamma < \tau) > 0$. Hence there exists a countable ordinal β depending on x such that $P^x(T_\beta = \tau) = 1$. Now from (b), $h(x) = P_\tau^A h(x)$ and since $x \in G$ is arbitrary, $h \in \mathcal{R}(G)$. $\square$

Here is the first application of 6.2. It extends Proposition 4.14 in [CZ95].

Theorem 6.3. *Let G, τ, μ, h and A be as in the first two sentences of Proposition 6.2. Let G be a countable union of sets $G_j \in \mathcal{O}$ with $h \in \mathcal{R}(G_j)$ for $j \ge 1$. Then $h \in \mathcal{R}(G)$.*

Proof. Let $D_n = G_n \setminus \bigcup_{j<n} G_j$. Then $\bigcup D_n = G$. Let $T = \tau(G_n)$ if $X_0 \in D_n$ and $T = 0$ if $X_0 \notin G$. Since $h \in \mathcal{R}(G_n)$ for $n \ge 1$ and $A^n \leftrightarrow 1_{G_n}\mu$ agrees with A on $[0, \tau(G_n)]$, it follows that T has the properties in 6.2. Consequently $h \in \mathcal{R}(G)$. $\square$

Corollary 6.4. *Let $G \in \mathcal{O}$, $\tau = \tau_G$, $1_G\mu \in \mathcal{S}(G)$ and $A \leftrightarrow 1_G\mu$. Let $h \in \mathcal{H}_f^\mu(G)$ and suppose that h is μ-regular on G. Assume that either X has no holding points or that $\mu^+(G) = 0$. Then $h = P_\tau^A h$ q.e. on G.*

Proof. By 5.8 and the fact that μ is smooth on G, there exists a μ-integrable decomposition (G_n) of G with $h = P_{\tau(G_n)}^A h$ on G_n. If $D = \bigcup G_n$, then 6.3 implies that $h = P_{\tau(D)}^A h$ on D. But $G \setminus D$ is m-polar so $P^x(T_{G\setminus D} < \infty) = 0$ for q.e. $x \in G$. Therefore $P_{\tau(D)}^A h = P_{\tau(G)}^A h$ q.e. on G, so $h = P_{\tau(G)}^A h$ q.e. on G. $\square$

Remark. Corollary 6.4 is close to optimal since one could not expect to have $h = P^A_{\tau(G)}h$ q.e. on G unless $t \to A_t$ is well-defined and finite on $[0, \tau_G[$, P^x a.s. for q.e. $x \in G$ which is equivalent to $1_G\mu \in \mathcal{S}(G)$.

The next result is a version of the mean value property approach to harmonic functions in the present context. It should be compared with Theorem 4.15 in [CZ95] and Theorem 2.2 in [CS98]. See also [H96] for a discussion of the restricted mean value property in classical potential theory.

Theorem 6.5. *Let $G \in \mathcal{O}$, $\tau = \tau_G$, $1_G\mu \in \mathcal{S}(G)$ and $A \leftrightarrow 1_G\mu$. Suppose $h : E \to \mathbb{R}$ is μ-regular on G, in particular $P^A_\tau|h| < \infty$ on G. Let $a : G \to]0, \infty[$ be nearly Borel and define*

$$\tau_x = \inf\{t > 0 : d(X_0, X_t) > a(x)\} \wedge \tau \tag{6.6}$$

where d is a metric on E compatible with the topology of E. If $h(x) = P^A_{\tau_x}h(x)$ for each $x \in G$, then $h = P^A_\tau h$ on G. If, in addition, X has no holding points or $\mu^+(G) = 0$, then $h \in \mathcal{H}^\mu_f(G)$.

For the proof we begin with the following lemma.

Lemma 6.7. *Let G, μ, h and A be as in the first two sentences of Theorem 6.5. Suppose that for each $x \in G$ there exists a stopping time T_x such that, $T_x \le \tau$, $P^x(T_x > 0) = 1$ and $h(x) = P^A_{T_x}h(x)$. If for each $t > 0$, $\{(x, \omega) : T_x(\omega) < t\} \in \mathcal{E}^n \times \mathcal{F}_t$, then the conclusions of Theorem 6.5 hold.*

Proof. Define $T(\omega) = T_{X_0(\omega)}(\omega)$ if $X_0(\omega) \in G$ and $T(\omega) = 0$ if $X_0(\omega) \notin G$. It is immediate that the assumptions on the family $\{T_x, x \in G\}$ imply that T satisfies the hypotheses of Proposition 6.2—recall the filtration $(\mathcal{F}_t)$ is right continuous. $\square$

Therefore in order to establish Theorem 6.5 it suffices to show that the τ_x defined in (6.6) satisfy the hypotheses of 6.7. Clearly the only thing that needs to be checked is the joint measurability of $\{(x, \omega) : \tau_x(\omega) < t\}$. If $a \in \mathbb{R}^+$, let $T_a = \inf\{t : d(X_0, X_t) > a\}$. Fix $t > 0$. Then $\{T_a < t\} = \bigcup_{r<t}\{d(X_0, X_r) > a\}$ where the union is over all rationals $r \in]0, t[$. Let $\Lambda(a, r) = \{d(X_0, X_r) > a\}$. For each a, $\Lambda(a, r) \in \mathcal{F}_r \subset \mathcal{F}_t$ while for each fixed ω, $a \to 1_{[0, d(X_0, X_r)[}(a)$ is right continuous in a. Consequently

$$\{(a, \omega) : \omega \in \Lambda(a, r)\} \in \mathcal{B} \times \mathcal{F}_r \subset \mathcal{B} \times \mathcal{F}_t$$

where $\mathcal{B}$ denotes the Borel σ-algebra of $[0, \infty[$. Hence $\{(a, \omega) : T_a(\omega) < t\} = \bigcup_{r<t}\{(a, \omega) : \omega \in \Lambda(a, r)\}$ is in $\mathcal{B} \times \mathcal{F}_t$. Since $x \to a(x)$ is nearly Borel it follows that $\{(x, \omega) : T_{a(x)}(\omega) < t\} \in \mathcal{E}^n \times \mathcal{F}_t$. It is now clear that $\tau_x = T_{a(x)} \wedge \tau$ satisfies the hypothesis of 6.7 completing the proof of Theorem 6.5. $\square$

Remark. The fact that d is a metric played a very minor role in the proof of (6.5). In fact if $\rho : E \times E \to \mathbb{R}^+$ is $\mathcal{E}^n \times \mathcal{E}^n$ measurable, $y \to \rho(x, y)$ is finely continuous for each $x \in E$ and $\rho(x, x) = 0$, then (6.5) holds with

$$\tau_x = \inf\{t > 0 : \rho(X_0, X_t) > a(x)\} \wedge \tau.$$

For example if $f \ge 0$ is a finite finely continuous function and $\rho(x, y) = |f(x) - f(y)|$.

We can now state and solve a "Dirichlet problem" for $\mathcal{H}^\mu_f(G)$.

Theorem 6.8. *Let $G \in \mathcal{O}$, $\tau = \tau(G)$ and $1_G\mu \in \mathcal{S}(G)$ with $A \leftrightarrow 1_G\mu$. Suppose that either X has no holding points or $\mu^+(G) = 0$. Let $g : G^c \to \mathbb{R}$ with $P_\tau^A|g| < \infty$ on G. Define $h = P_\tau^A g$ on G and $h = g$ on G^c. Then $h \in \mathcal{H}_\mu^f(G)$. If, in addition, h is μ-regular on G, then h is the quasi-unique element of $\mathcal{H}_f^\mu(G)$ that is μ-regular on G and equals g q.e. on G^c. Here quasi-unique means any other such $\tilde{h}$ equals h q.e. on G.*

Proof. It follows from 5.8 that $h \in \mathcal{H}_f^\mu(G)$ and then from 6.4 that $h = P_\tau^A h$ q.e. on G. If $\tilde{h} \in \mathcal{H}_f^\mu(G)$ is μ-regular on G, then $\tilde{h} = P_\tau^A \tilde{h}$ q.e. on G by 6.4. But $\tilde{h} = g$ q.e. on G^c and so $P_\tau^A \tilde{h} = P_\tau^A g$ q.e. on G. Combining these statements gives $h = \tilde{h}$ q.e. on G. □

Remark. It follows from 5.10 that if X is special, then h as defined in the statement of 6.8 is automatically μ-regular on G.

The next corollaries contain special cases of particular importance.

Corollary 6.9. *Let X be a Hunt process on a locally compact Hausdorff space E with a countable base. Let d be a metric on E compatible with the topology of E. Let $G \subset E$ be open, $\tau = \tau_G$ and $h : E \to \mathbb{R}$ be bounded and continuous on $\bar{G}$. Let $1_G\mu \in \mathcal{S}(G)$ and $A \leftrightarrow 1_G\mu$. Assume that on G, $E^\cdot(e^{A_\tau^*}) < \infty$ and $P_\tau^A|h| < \infty$ where A_τ^* is defined in 5.12. Let $a : G \to]0,\infty[$ be nearly Borel and such that $a(x) < d(x, G^c)$. Suppose $\tau < \infty$ a.s. Define*

$$\tau_x = \inf\{t > 0 : d(X_0, X_t) > a(x)\}.$$

Then h is μ-regular on G. If $h(x) = P_{\tau_x}^A h(x)$ for each $x \in G$, then $h = P_\tau^A h$ on G.

Proof. The μ-regularity follows from 5.13. The conclusion is then an immediate consequence of 6.5. Note that $P^x(\tau_x \le \tau) = 1$ for $x \in G$ and that it is not assumed that $\bar{G}$ is compact. □

Corollary 6.10. *Let X, E and d be as in 6.9. Let $G \subset E$ be open and $h : E \to \mathbb{R}$ be continuous on G. Let μ be Radon on G; i.e. $|\mu|(K) < \infty$ for compact $K \subset G$. Assume either that X has no holding points or that $\mu^+(G) = 0$. Suppose that if D is open with compact closure $\bar{D} \subset G$, then $\tau_D < \infty$ a.s., $P_{\tau(D)}^A|h| < \infty$ and $E^\cdot[e^{A^*(\tau(D))}] < \infty$. Let (r_n) be a sequence of strictly positive numbers with $r_n \to 0$ having the property that for each $x \in G$ there exists a subsequence $(r_n(x))$ of (r_n) such that if $r_n(x) < d(x, G^c)$, then $h(x) = P_{\sigma(r_n(x))}^A h(x)$ where $\sigma(r) = \inf\{t : d(X_0, X_t) > r\}$. Under these hypotheses $h \in \mathcal{H}^\mu(G)$.*

Proof. Let D be open with $\bar{D}$ compact and $D \subset \bar{D} \subset G$. Since $|\mu|(\bar{D}) < \infty$, $1_D\mu \in \mathcal{S}(D)$. Let

$$D_n := \{x \in D : r_n < d(x, D^c) \text{ and } h(x) = P_{\sigma(r_n)}^A h(x)\}.$$

Since $r_n \to 0$ the hypotheses imply that $D = \bigcup D_n$. If $x \in D_n \setminus \bigcup_{k<n} D_k$ define $a(x) = r_n$. Then the hypotheses of 6.9 are satisfied relative to D and so $h = P_{\tau(D)}^A h$ on D. Then 5.8 implies that $h \in \mathcal{H}^\mu(D)$. But G is a countable (increasing) union of such sets D and $h \in \mathcal{H}^\mu(G)$. □

These corollaries say something non-trivial even in the most classical situation of Brownian motion on $\mathbb{R}^d$. Of course, there are much more refined statements in that case. See, for example, [H96] and the references therein.

There are analogous results for the representation $h = P_\tau h + U_A^\tau h$ which we briefly sketch. The following companion to 5.10 justifies the hypotheses of 6.12.

Proposition 6.11. *Let $D \in \mathcal{O}$, $1_D\mu \in \mathcal{S}(D)$ and $A \leftrightarrow 1_D\mu$. Let $\tau = \tau_D$. Let $u : E \to \mathbb{R}$ and suppose that on D, $P_\tau|u| < \infty$, $U_{|A|}^\tau|u| < \infty$ and $u = P_\tau u + U_A^\tau u$. Then u is regular on D.*

Proof. Since $U_A^\tau u$ is finely continuous of D, 5.2 implies that u is finely continuous on D. Let $Y_\tau = u(X_\tau)1_{\{\tau<\infty\}} + \int_0^\tau u(X_t)\,dA_t$. Then it follows from 3.6 in [G99b] that if T is a stopping time with $T \le \tau$, then $E^x(Y_\tau \mid \mathcal{F}_T) = Y_T$ for $x \in D$. Hence $E^\cdot(|Y_T|) \le E^\cdot(|Y_\tau|) < \infty$ on D and so $P_T|u| < \infty$ and $U_{|A|}^T|u| < \infty$ on D. If $T_n \uparrow T \le \tau$, $U_A^{T_n}u \to U_A^T u$ and it follows as in the proof of 5.10 that $P_{T_n}u \to P_T u$ on D . □

The following result is proved similarly to the proof of 6.2 using transfinite induction. The argument when β has an immediate predecessor is somewhat more complicated, but presents no essential difficulty. Note that h is only required to be regular and not μ-regular. Once it is established results analogous to 6.3 and 6.5 are easily proved. We leave their formulation to the interested reader.

Theorem 6.12. *Let $G \in \mathcal{O}$, $\tau = \tau_G$, $1_G\mu \in \mathcal{S}(G)$ and $A \leftrightarrow 1_G\mu$. Let $h : E \to \mathbb{R}$ be regular on G. Suppose $U_{|A|}^\tau|h| < \infty$ on G. If there exists a stopping time T such that $T \le \tau$, $T = 0$ on $\{X_0 \notin G\}$ and on G, $P^\cdot(T > 0) = 1$ and $h = P_T h + U_A^T h$, then $h = P_\tau h + U_A^\tau h$ on G. In particular $h \in \mathcal{H}_f^\mu(G)$.*

7. Concluding Remarks

There is an annoying exceptional set in our definition of finely μ-harmonic functions. This is necessary because of the exceptional set in the definition (3.9–iii) of a μ-integrable decomposition which in turn comes from the exceptional set in Proposition 3.6. So finally it is due to the exceptional set in the definition of a CAF of (X, τ). It is of interest and importance to know conditions that guarantee that these exceptional sets are empty. This is certainly the case if $\mu = 0$. On the other hand if m is a *reference* measure, then assuming a somewhat stronger condition on μ will ensure that all of the exceptional sets are empty. Recall that m is a reference measure provided the resolvent $U^q(x, \cdot) << m$ for all x for one, and hence all, $q \ge 0$. Under this assumption m-polar and m-semipolar reduce to polar and semipolar. Also q-excessive functions are Borel measurable for $q \ge 0$ and if two q-excessive functions agree a.e., they are identical.

In order to describe our result we need the resolvent $(\hat{U}^q)_{q\ge 0}$ of the moderate Markov dual process $\hat{X}$ of X relative to m. See, for example, [F87] or [G99a]. It follows that if μ is a positive measure not charging polar sets, then $\mu\hat{U}^q << m$ and if $\mu\hat{U}^q$ is σ-finite, then it has a unique q-excessive density v^q relative to m. Since $v^q < \infty$ a.e., $\{v^q = \infty\}$ is polar. If X and $\hat{X}$ are in strong duality relative to m with resolvent density $u^q(x, y)$ as in [BG68] for example, then

$$v^q(x) = \int u^q(x, y)\mu(dy) = U^q\mu(x).$$

Under strong duality this makes sense for any positive measure μ. We are now able to introduce the relevant definitions. These are patterned after those in [FOT94].

Definitions 7.1. (i) A strict PCAF or CAF of X is one with empty exceptional set.

(ii) If G is a finely open *Borel* set a strict nest (G_n) for G is an increasing sequence of finely open Borel subsets of G with $\tau(G_n) \uparrow \tau := \tau(G)$ a.s. P^x for all x. If $G = E$ we just say a strict nest $(\tau(E) = \zeta)$.

(iii) $\mu \in \mathcal{S}_0^+$ is strictly smooth on a finely open Borel set G provided there exists a strict nest (G_n) for G with $\mu(G_n) < \infty$ for each n and a $q \geq 0$ such that if $\mu_n = \mu|_{G_n}$ then $\mu_n \hat{U}^q$ is σ-finite and the q-excessive version of $d(\mu_n \hat{U}^q)/dm$ is everywhere finite for each n.

(iv) $\mu \in \mathcal{S}_0$ is strictly smooth provided $|\mu|$ is strictly smooth or equivalently μ^+ and μ^- are strictly smooth.

Remarks. Since μ_n in (7.1–iii) is finite, $\mu_n \hat{U}^q$ is automatically σ-finite when $q > 0$. In (7.1–iii) the q-excessive version u_n^q of $d(\mu_n \hat{U}^q)/dm$ is always finite off a polar set. Thus the crucial condition is that it be finite *everywhere*. In particular if $\mu_n \hat{U}^q \leq c_n m$ where $c_n < \infty$, then u_n^q is bounded by c_n.

The key result is contained in the next theorem. It is proved by arguments similar to those used on pages 194–196 of [FOT94]. Under strong duality it goes back in essence to Revuz' original paper [Re70]. For the convenience of the reader we shall give a proof in the appendix.

Theorem 7.2. *Assume that m is a reference measure and let $\mu \in \mathcal{S}_0^+$. Then μ is the Revuz measure of a (unique) strict PCAF, A, provided μ is strictly smooth. Conversely if A is a strict PCAF then the Revuz measure ν_A of A is strictly smooth and if $q > 0$ ($q = 0$ if X is transient) the strict nest may be chosen so that $E = \bigcup G_n$ and $\mu_n \hat{U}^q \leq c_n m$ for each n where $\mu_n = \mu|_{G_n}$ and $c_n < \infty$.*

Suppose in this paragraph that m is an excessive reference measure. If $G \in \mathcal{O} \cap \mathcal{E}$ then $G_p = \{x : E^x(e^{-\tau(G)}) > 0\}$ is a finely open Borel set and hence (X, τ_G) is a Borel right process with $m|_G = m|_{G_p}$ being an excessive reference measure. Therefore 7.2 maybe be applied directly to (X, τ_G). Thus we shall say that $G \in \mathcal{O} \cap \mathcal{E}$ has a *strict* μ-integrable decomposition provided we modify (3.9–iii) by requiring $G_n \in \mathcal{O} \cap \mathcal{E}$ for each n and $G = \bigcup G_n$. Define Λ_G^* by replacing μ-integrable by strict μ-integrable in Definition 4.4. Finally put $\mu \in \mathcal{S}_{loc}^*$ provided $E = \bigcup G_n$ where each $G_n \in \mathcal{O} \cap \mathcal{E}$ and $\mu_n = \mu|_{G_n}$ is strictly smooth on G_n for each n. If in Sections 5 and 6 we fix $\mu \in \mathcal{S}_{loc}^*$ and replace $\mathcal{O}$ by $\mathcal{O} \cap \mathcal{E}$, Λ_G by Λ_G^* and μ-integrable by strictly μ-integrable, then all of the results are valid without exceptional sets.

In $\mathbb{R}^d$ or more generally on a Riemannian manifold the equation $\Delta u + u\mu = 0$ is usually interpreted in the sense of distributions. It is of interest to note the connection with the definition in Section 5. For simplicity let X be Brownian motion on $\mathbb{R}^d$. Then $\Lambda := -\frac{1}{2}\Delta$ is the "formal" generator of X. Let $G \subset \mathbb{R}^d$ be open and greenian; i.e. G arbitrary if $d \geq 3$ and G^c non-polar if $d = 1$ or 2. Let $g(x, y)$ be the Green function of G. Suppose that $\mu \in \mathcal{S}_0$, $|\mu|$ is Radon on G and if H is open with compact closure in G, then $\int_H g(x, y)|\mu|(dy)$ is bounded. This implies that μ is strictly smooth on G since there is a strict nest for G consisting of sets H as in the preceding sentence. Let $A \leftrightarrow 1_G\mu$. Suppose that $H \subset \bar{H} \subset G$ with H open and $\bar{H}$ compact and let $\tau = \tau(H)$. Assume that $u : G \to \mathbb{R}$ and that $\Lambda u + u\mu = 0$ on G in the sense of distributions. In order that $u\mu$ define a distribution on G, $\int_H u d\mu$ must be well-defined for all such H. To ensure this we shall suppose that u is finely continuous and locally bounded on G. If $\varphi \in C_c^\infty(H)$—the C^∞

functions with compact support in H—then writing $(f,g) = \int fg dm$ where m is Lebesgue measure on $\mathbb{R}^d$ we have $(\Lambda\varphi, u) + \int \varphi u \, d\mu = 0$. But $C_c^\infty(H)$ is contained in the domain of the generator of (X,τ) which coincides there with $\Lambda = -\frac{1}{2}\Delta$. Therefore $\varphi = -U^\tau \Lambda\varphi$. In the present situation $U_A^\tau f = \int_H g_H(\cdot, y)\mu(dy)$ where g_H is the Green function for (X,τ). Hence

$$\begin{aligned} 0 &= (\Lambda\varphi, u) - \int_H U^\tau(\Lambda\varphi) u \, d\mu \\ &= (\Lambda\varphi, u) - (\Lambda\varphi, U_A^\tau u) = (\Lambda\varphi, u - U_A^\tau u), \end{aligned}$$

where the second equality follows from the symmetry of g_H. It follows that there is an harmonic function h on H with $u - U_A^\tau u = h$ a.e. and hence everywhere on H because u and $U_A^\tau u$ are finely continuous. Of course

$$U_{|A|}^\tau |\mu| = \int_H g_H(\cdot, y)|u|(y)|\mu|(dy) < \infty.$$

Let J be open with compact closure $\bar{J} \subset H$ and let $\sigma = \tau(J)$. Then by the usual Poisson representation $h = P_\sigma h = P_\sigma u - P_\sigma U_A^\tau u$ on J. Therefore $u = P_\sigma u + U_A^\sigma u$ and since H, and hence G, is a countable union of such sets J, $u \in \mathcal{H}_f^\mu(G)$ and if u is continuous, $u \in \mathcal{H}^\mu(G)$.

Finally the definitions of Λ_G and the elements of $\mathcal{H}_f^\mu(G)$ may be cast in terms of martingales. This is hinted at in the proofs of 5.2, 5.10 and 6.6. It is spelled out in more detail in [G99b]. In particular Theorem 3.9 in [G99b] gives the equivalence of the current definition and one in terms of martingales. The interested reader is referred to the discussion there.

Appendix

The basic step in proving Theorem 7.2 is the next result which should be compared with Theorem 5.1.6 in [FOT94]. The notation is that of Section 7.

Theorem A.1. *Suppose that m is an excessive reference measure. Let $\mu \in \mathcal{S}_0^+$ with $\mu(E) < \infty$. Suppose that for some $q \geq 0$, $\mu\hat{U}^q$ is σ-finite. Then $\mu\hat{U}^q << m$. Let u^q be the unique q-excessive version of $d(\mu\hat{U}^q)/dm$. If $u^q < \infty$ everywhere, then there exists a unique strict PCAF, A, such that $u^q = u_A^q$ everywhere and μ is the Revuz measure of A. Here $u_A^q := E^\cdot \int_0^\infty e^{-qt} \, dA_t$ and more generally $U_A^q f := E^\cdot \int_0^\infty e^{-qt} f(X_t) \, dA_t$ whenever the integrals exist.*

Proof. For simplicity we shall write the proof when $q = 0$. To show $\mu\hat{U} << m$, let $A \in \mathcal{E}$ with $m(A) = 0$. Let $B = \{x : \hat{U}(x, A) > 0\}$. Since m is excessive relative to $(\hat{U}^q)$, $qm\hat{U}^q(A) \leq m(A) = 0$. Therefore $0 = m\hat{U}^q(A) \uparrow m\hat{U}(A)$ as $q \downarrow 0$; that is $\hat{U}(\cdot, A) = 0$ a.e. and hence by (2.10) of [G99a], $\hat{U}(\cdot, A) = 0$ q.e. But μ doesn't charge m-polar sets and so $\mu\hat{U}(A) = 0$. Thus $\mu\hat{U} << m$. If $\mu\hat{U}$ is σ-finite, then it is a coexcessive measure and hence has an excessive density u which is uniquely determined since m is a reference measure.

We now suppose that $u < \infty$ everywhere. Since μ is finite and, hence smooth there exists a PCAF, B, with Revuz measure μ. By (3.7) of [G99a], if $u_B := E^\cdot(B_\infty)$ one has $u_B m = \mu\hat{U}$ and so $u_B = u$ a.e. Let N be the polar exceptional set for B. Then u_B is excessive for X restricted to $E \setminus N$ and it follows that $u_B = u$ on $E \setminus N$. Let Λ be the defining set for B. Then $P^x(\Lambda) = 1$ for $x \notin N$. Let

$\varepsilon_n = \varepsilon(n) \downarrow\downarrow 0$ and put $\Lambda_0 = \bigcap_n \theta^{-1}_{\varepsilon(n)}\Lambda$. But $\theta_t\Lambda \subset \Lambda$ for all $t \geq 0$. Thus if $t > 0$ and $\varepsilon_n < t$, $\theta_t\Lambda_0 \subset \theta_t\theta^{-1}_{\varepsilon_n}\Lambda \subset \theta_{t-\varepsilon_n}\Lambda \subset \Lambda$. Also $\omega \in \Lambda_0$ if and only if $\theta_{\varepsilon_n}\omega \in \Lambda$ for all n. Let $\omega \in \Lambda_0$. Then $\theta_{\varepsilon_n}\theta_t\omega = \theta_t\theta_{\varepsilon_n}\omega \in \Lambda$ for each n. Thus $\theta_t\Lambda_0 \subset \Lambda_0$; hence $\theta_t\Lambda_0 \subset \Lambda \cap \Lambda_0$. For each $x \in E$, $P^x(\theta^{-1}_{\varepsilon_n}\Lambda) = E^x[P^{X(\varepsilon_n)}(\Lambda)] = 1$ since $\varepsilon_n > 0$ and N is polar. Thus $P^x(\Lambda_0) = 1$ for all x. Define for $t > \varepsilon_n$ and $\omega \in \Omega$

$$B^n_t(\omega) = B_{t-\varepsilon_n}(\theta_{\varepsilon_n}\omega).$$

If $\omega \in \Lambda_0$, $n > m$ and $t > \varepsilon_m$, then

$$\begin{aligned} B^n_t(\omega) &= B_{t-\varepsilon_m+\varepsilon_m-\varepsilon_n}(\theta_{\varepsilon_n}\omega) \\ &= B_{t-\varepsilon_m}(\theta_{\varepsilon_m}\omega) + B_{\varepsilon_m-\varepsilon_n}(\theta_{\varepsilon_n}\omega). \end{aligned}$$

Hence $B^n_t(\omega) \geq B^m_t(\omega)$ on $t > \varepsilon_m$. Define

$$A_t(\omega) = \lim_n B^n_t(\omega), \qquad t > 0, \omega \in \Lambda_0.$$

If $t > \varepsilon_m$ and $B^m_t(\omega) < \infty$, $B^n_t(\omega) - B^m_t(\omega) = B_{\varepsilon_m-\varepsilon_n}(\theta_{\varepsilon_m}\omega)$. Thus if for some $t > 0$, $A_t(\omega) < \infty$, then $B_{\varepsilon_m-\varepsilon_n}(\theta_{\varepsilon_m}\omega) \to 0$ as $n, m \to \infty$ and so $B^n_s(\omega) \to A_s(\omega)$ uniformly on $]0, t]$. In particular A is continuous on $]0, t]$. If $x \in E \setminus N$, $E^x(B_\infty) = u_B(x) = u(x) < \infty$, hence for $t > 0$

$$E^x(B_t) = E^x(B_\infty) - E^x(B_\infty - B_t) = u(x) - P_t u(x).$$

Thus for all x and $t > \varepsilon_n$

$$E^x(B^n_t) = E^x[E^{X(\varepsilon_n)}(B_{t-\varepsilon_n})] = P_{\varepsilon_n}u(x) - P_t u(x).$$

Let $n \to \infty$, so that $B^n_t \uparrow A_t$ on Λ_0 and hence a.s., to obtain $E^x(A_t) = u(x) - P_t u(x) < \infty$. Therefore $E^x(A_\infty) \leq u(x) < \infty$ for all x. Define $A_t(\omega) = 0$ for all t if $\omega \notin \Lambda_0$. It follows that $t \to A_t$ is finite and continuous on $]0, \infty]$ a.s.—i.e. a.s. P^x for all x. Also letting $t \downarrow 0$, $E^x(A_{0+}) = 0$ for all x.

If $s, t > 0$ and $\omega \in \Lambda_0$, then for $s > \varepsilon_k$

$$\begin{aligned} A_{t+s}(\omega) &= \lim_n B_{t+s-\varepsilon_n}(\theta_{\varepsilon_n}\omega) \\ &= \lim_n [B_{t-\varepsilon_n}(\theta_{\varepsilon_n}\omega) + B_s(\theta_t\omega)] \\ &= A_t(\omega) + B_{s-\varepsilon_k}(\theta_{\varepsilon_k}\theta_t\omega) + B_{\varepsilon_k}(\theta_t\omega). \end{aligned}$$

Now $\theta_t\omega \in \Lambda_0 \cap \Lambda$ and so $B_{s-\varepsilon_k}(\theta_{\varepsilon_k}\theta_t\omega) \to A_s(\theta_t\omega)$ and $B_{\varepsilon_k}(\theta_t\omega) \to 0$ since $B_{0+} = 0$ on Λ. Consequently

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega); \qquad t, s > 0, \omega \in \Lambda_0. \tag{A.2}$$

Define

$$\Lambda_1 = \{\omega \in \Lambda_0; t \to A_t(\omega) \text{ is finite and continuous on }]0, \infty] \text{ and } A_{0+}(\omega) = 0\}.$$

Then $\theta_t\Lambda_1 \subset \Lambda_1$ and $P^x(\Lambda_1) = 1$ for all x. Define $A_0 = 0$. Then $A_0 = A_{0+}$ on Λ_1 and (A.2) holds for $s, t > 0$ and $\omega \in \Lambda_1$. Therefore A is a PCAF with defining set Λ_1 and empty exceptional set.

Finally if $x \in E \setminus N$

$$P_t u(x) = P_t u_B(x) = E^x(B_\infty - B_t) \to 0$$

as $t \to \infty$ since $E^x(B_\infty) < \infty$. But $E^x(A_t) = u(x) - P_t u(x)$ and so $u_A = u = u_B$ on $E \setminus N$. Therefore $A = B$ a.s. P^x for $x \in E \setminus N$. But $m(N) = 0$ and so if $f \geq 0$

$$\nu_A(f) = \lim_{t \to 0} \frac{1}{t} E^m \int_0^t f(X_s)\, dAs = \lim_{t \to 0} \frac{1}{t} E^m \int_0^t f(X_s)\, dB_s = \nu_B(f) = \mu(f).$$

Therefore $\nu_A = \mu$. □

We now turn to the proof of (7.2). Suppose μ is strictly smooth with strict nest (G_n). Let $\mu_n = \mu|_{G_n}$. Then there exists $q \geq 0$ such that $\mu_n \hat{U}^q = u_n m$ where u_n is the unique q-excessive version of $d(\mu_n \hat{U}^q)/dm$ and $u_n < \infty$ everywhere. By A.1 there exists a strict PCAF, A^n, with Revuz measure μ_n and such that

$$u^q_{A_n} = E^{\cdot} \int_0^\infty e^{-qt}\, dA^n_t = u_n.$$

If $f \geq 0$ is bounded then $(f * A^n)_t := \int_0^t f(X_s)\, dA^n_s$ is a PCAF with Revuz measure $f\mu_n$. Since $\mu_n(G_n^c) = 0$ it follows that $1_{G_n} * A^n = A^n$. For $f \geq 0$ and $k < n$ using (3.7) of [G99a] for the first and third equalities one has

$$(f, U^q_{A^n} 1_{G_k}) = \int_{G_k} \hat{U}^q f\, d\mu_n = \int_{G_k} \hat{U}^q f\, d\mu_k = (f, U^q_{A^k} 1_{G_k}) = (f, u^q_{A^k}).$$

Therefore $u^q_{A^k} = U^q_{A^n} 1_{G_k}$ a.e. and hence everywhere because m is a reference measure. Consequently $A^k = 1_{G_k} * A^n$ for $k < n$; in particular $A^k_t = A^n_t$ on $[0, \tau_{G_k}[$ and $A^k_t \leq A^n_t$ on $[0, \infty]$. But $\tau_{G_n} \uparrow \zeta$ a.s. and so $A_t := \lim_n A^n_t$ exists and is finite and continuous on $[0, \zeta[$ a.s. Therefore A is a PCAF with defining set $\{\lim \tau_{G_n} = \zeta\}$ and empty exceptional set. Let ν_A be the Revuz measure of A. Since A^n and A agree on $[0, \tau_{G_n}]$ it follows from (2.22) of [FG88], that $1_{G_n} \nu_A = 1_{G_n} \nu_{A^n} = \mu_n$, and letting $n \uparrow \infty$ we obtain $\nu_A = \mu$.

Conversely because excessive functions are Borel measurable the argument in 3.6 of the present paper or (3.11a) of [FG96] shows that if A is a strict PCAF, then ν_A is strictly smooth and the nest (G_n) may be chosen so that $E = \bigcup G_n$ and $U^q_A 1_{G_n}$ is bounded for $q > 0$ or, when X is transient, for $q \geq 0$. But then

$$(1_{G_n} \nu_A) \hat{U}^q = U^q_A 1_{G_n} \cdot m \leq c_n \cdot m$$

where c_n is a bound for $U^q_{A^n} 1_{G_n}$. This completes the proof of 7.2. □

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